Frequency Response

Hanwool Jeong

hwjeong@kw.ac.kr

Capacitor

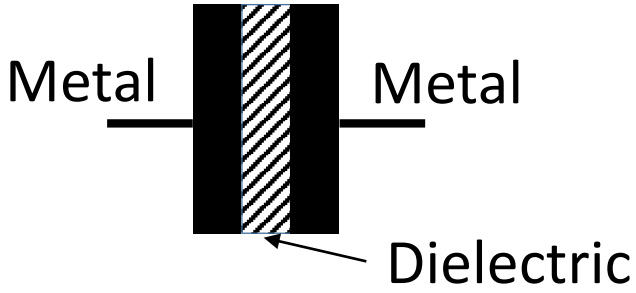
- Definition of a capacitor

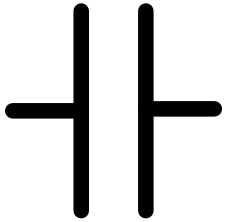
$$Q = CV \rightarrow I = C(dV/dt)$$

➔ To change voltage, current must flow into/from the capacitor.

➔ **It is impossible to abruptly change voltage across a capacitor**

- Typical shape of a capacitor



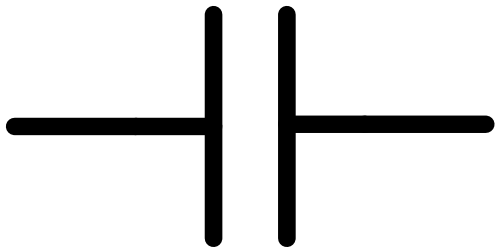


- Have ever asked question like:

➔ How is the current flow possible in a capacitor?

Current is Nothing but Charge Flow

- Charge cannot flow through the dielectric, but the charge flow seen at the outside is exactly as the charge flows into it.



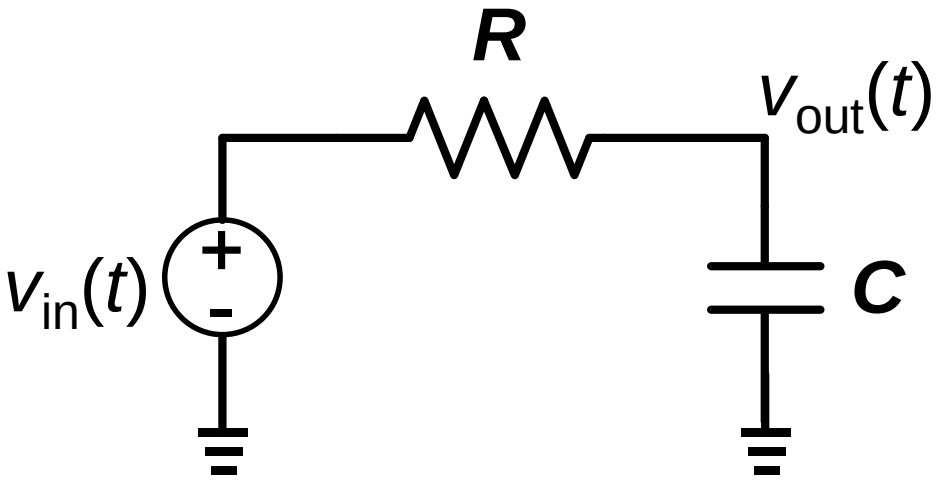
- This current is formed only during the voltage is changing.

➔No change in voltage means no current flow in a capacitor

➔Less change cap can be let open while being meaningful when signal changes frequently.

Forming Differential Equation

- Can you derive the differential eq. relating $v_{out}(t)$ to $v_{in}(t)$?



$$\frac{v_{in}(t) - v_{out}(t)}{R} = C\frac{dv_{out}(t)}{dt}$$

$$RC\frac{dv_{out}(t)}{dt} + v_{out}(t) = v_{in}(t)$$

Laplace Transform for Solving Differential Equation

- Unilateral Laplace transform is defined as

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

where s is complex number. They comprise a pair as

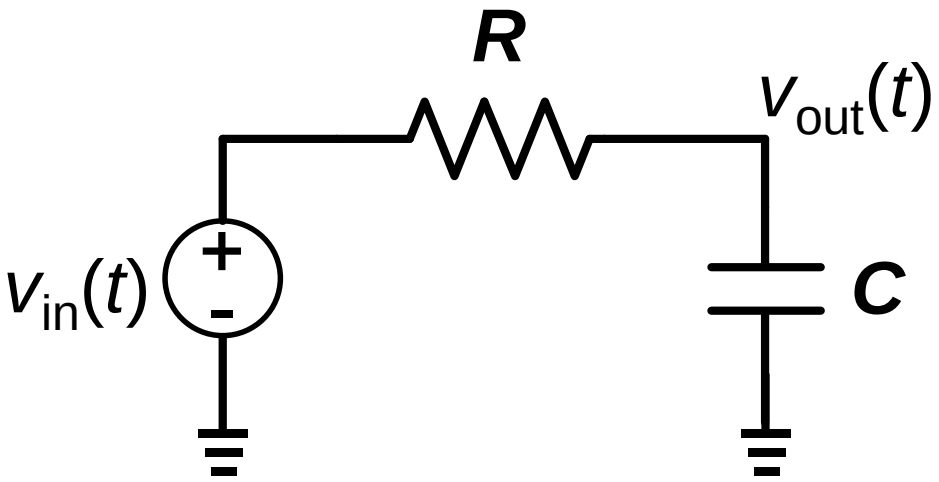
$$f(t) \leftrightarrow F(s)$$

- When $f(t)$ is differentiated in the time domain,

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_{0^-}^{\infty} \frac{df}{dt}e^{-st}dt \quad = s\int_0^\infty fe^{-st}dt + fe^{-st}\Big|_0^\infty = sF(s) - f(0^-)$$

Revisit RC Circuit

- If $v_{out}(0) = 0$,



$$\frac{v_{in}(t) - v_{out}(t)}{R} = C\frac{dv_{out}(t)}{dt}$$

$$\leftrightarrow \quad \frac{V_{in}(s) - V_{out}(s)}{R} = sCV_{out}(s) \quad = \frac{V_{out}(s)}{1/sC}$$

$$RC\frac{dv_{out}(t)}{dt} + v_{out}(t) = v_{in}(t)$$

$$\leftrightarrow \quad RsCV_{out}(s) + V_{out}(s) = V_{in}(s)$$

- Note that capacitor can be dealt with in the same manner with a resistor whose value is **1/sC** in the s domain.

AC Circuit; Sinusoidal Signal Input

- Sinusoidal input for linear circuit at the steady state. All the I-V signal in the circuit have the frequency of ω (rad/s).
- That is, any voltage and current can be represented as

$$v(t) = V_0 \cos(\omega t + \theta_V) \qquad i(t) = I_0 \cos(\omega t + \theta_I)$$

- Suppose the voltage across a cap is $v_C(t) = V_0 \cos(\omega t)$ then,

$$i_C(t) = C\frac{dv_C(t)}{dt} = -\omega C V_0 \sin(\omega t) = \omega C V_0 \cos(\omega t + 90°)$$

+ 90° in phase

×ωC in magnitude

- We can see the higher the frequency is, the larger the current flow in a capacitor is.

Relating to Laplace Transform

- Recall that

$$i_C(t) = C\frac{dv_C(t)}{dt} \quad \longleftrightarrow \quad I_C(s) = sCV_C(s)$$

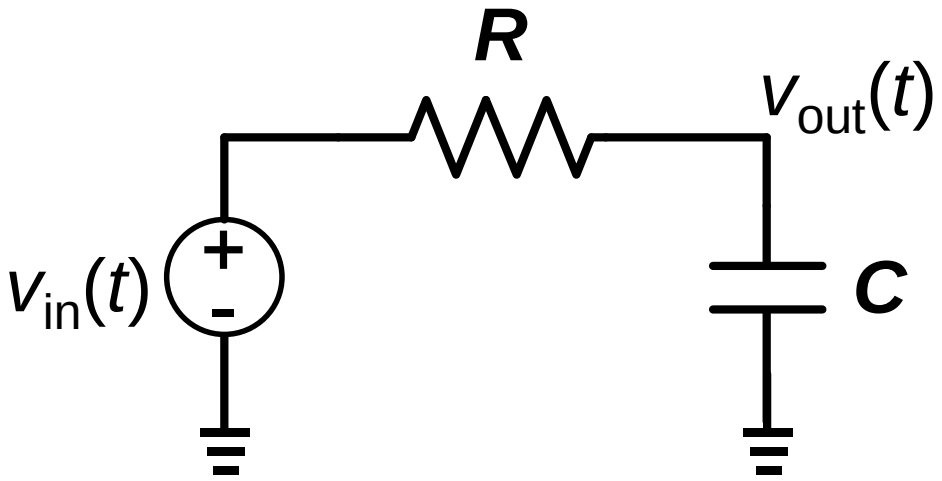
- On the other hand, for an AC circuit with sinusoidal input whose freq is ω, $v_C(t) = V_0\cos(\omega t)$ then

$$i_C(t) = \omega C V_0 \cos(\omega t + 90°) \quad \longleftrightarrow \quad I_C(s) = \omega C e^{+\frac{90°}{\omega}s} V_C(s)$$

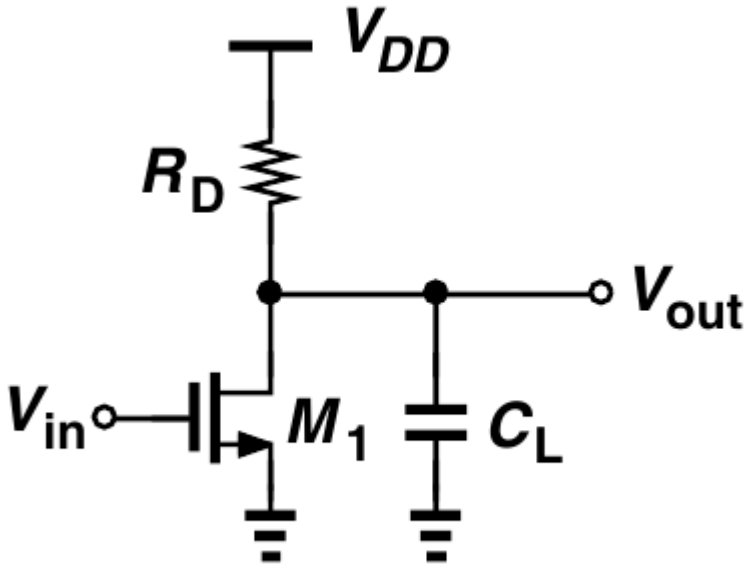
- That means **s = jω** and we can see the frequency response from it. Espeically, 1/sC = 1/jωC is called impedance of a cap.

What if Frequency Changes? (1)

- At low freq, C is nearly open ➔ Why?
- As freq ↑, decreased impedance of C results in a large $v_{out}(t)$.
  ➔ **At high frequency. we cannot ignore the cap**
- In general, we can analyze the circuit using the impedance concept $1/sC = 1/j\omega C$ for frequency response.



What if Frequency Changes? (2)



$$V_{out} = -g_m V_{in} \left(R_D || \frac{1}{C_L s} \right)$$

Transfer Function & Frequency Response

- The transfer function **H(s)** is output to input ratio in the s domain (Laplace transformed) when the initial conditions are zero.

$$H(s) = \frac{output(s)}{input(s)} = A_0 \frac{\left(1 + \frac{s}{\omega_{z1}}\right)\left(1 + \frac{s}{\omega_{z2}}\right)\ldots}{\left(1 + \frac{s}{\omega_{p1}}\right)\left(1 + \frac{s}{\omega_{p2}}\right)\ldots}$$

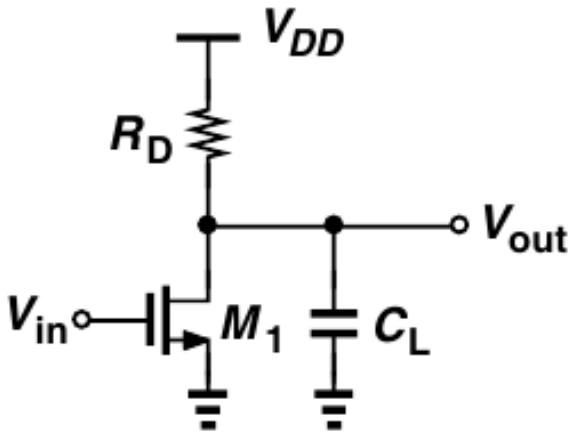
- ω_{zj} and ω_{pj} represent the zeros and poles of H(s), respectively.
- When input is x(t) = A cosωt, the output can be expressed as

$$y(t) = A|H(j\omega)|\cos[\omega t + \angle H(j\omega)],$$

- Frequency response can be represented with the magnitude and phase of H(s) when s=jω, that is H(jω)

Example 11.4

Determine the transfer function and frequency response of the CS stage shown in Fig. 11.5(a).

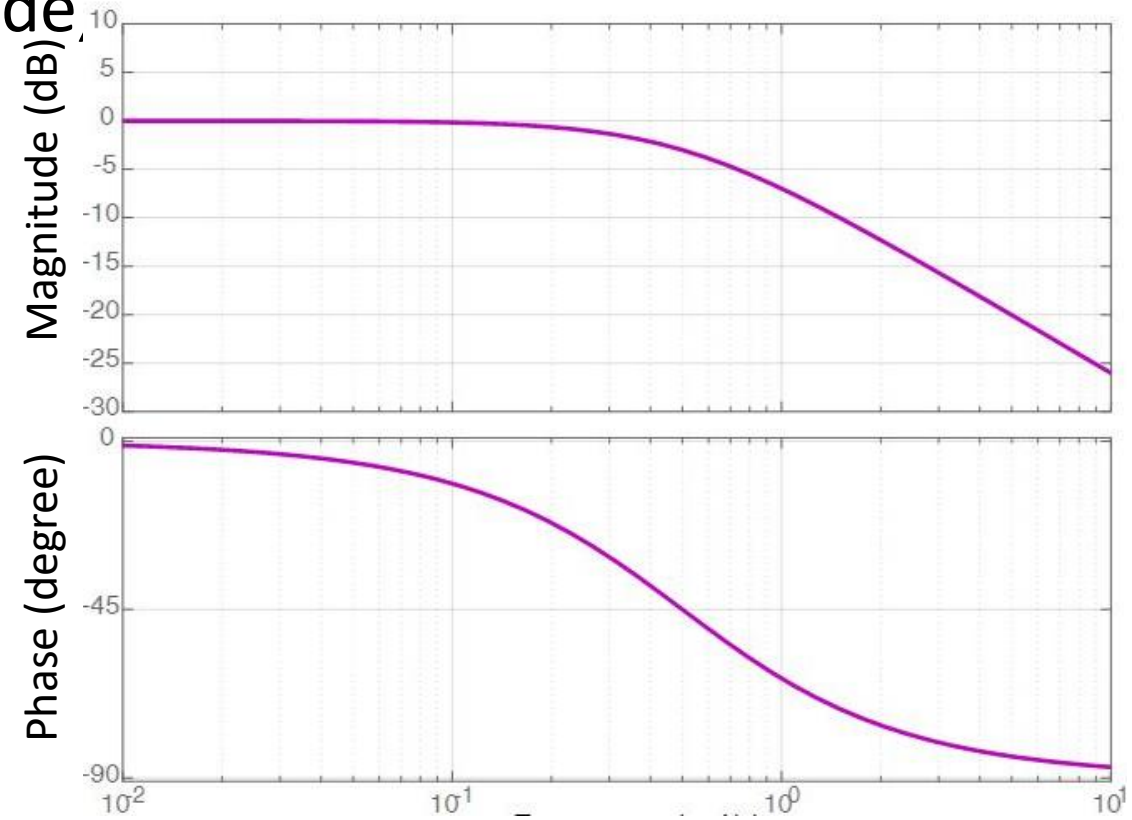


Bode Plot

- The Bode plot is a chart of **H(jω) in decibels and phase in degrees** versus the **logarithm of frequency**.
- The logarithm of the magnitude, the gain in dB:

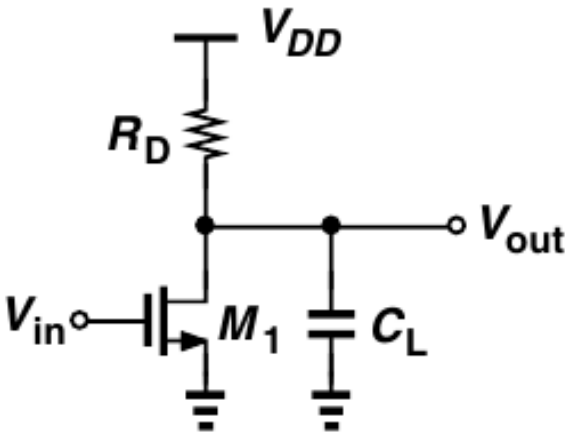
$$\text{logarithmic gain} = 20 \log_{10} \text{H}$$

Unit : decibel (dB)



Asymptotic Bode Plot

Example 11.7 Construct the Bode plot of $|H(j\omega)|$ for the CS stage



$$H(s) = \frac{V_{out}}{V_{in}}(s) = -g_m\left(R_D \| \frac{1}{C_L s}\right)$$

$$= \frac{-g_m R_D}{R_D C_L s + 1}.$$

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{g_m R_D}{\sqrt{R_D^2 C_L^2 \omega^2 + 1}}.$$

Bode's Rules

|H(jω)| can be approximated with the following rules:

- As ω passes each pole frequency, the slope of |H(jω)| decreases by 20 dB/dec. (A slope of 20 dB/dec simply means a tenfold change in H for a tenfold increase in frequency);
- • As ω passes each zero frequency, the slope of |H(jω)| increases by 20 dB/dec.

Association of Poles with Nodes

- The poles of H(s) can tell "speed bottleneck." of a circuit

➔ You should identify the poles intuitively.

- In the previous CS example, how was ω_p is given?
- Generally, if node j in the signal path exhibits a small signal resistance of Rj to ground and a capacitance of C_j to ground,

$$\omega_p = 1/R_jC_j$$

Example 11.8

Determine the poles of the circuit shown in Fig. 11.10. Assume $\lambda = 0$.

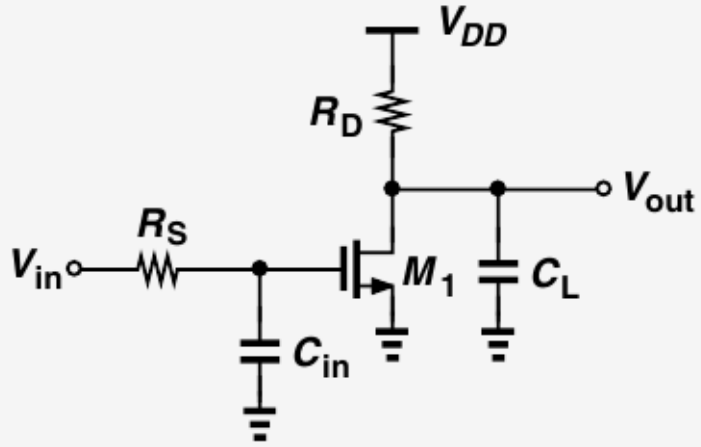


Figure 11.10

Example 11.9

Compute the poles of the circuit shown in Fig. 11.11. Assume $\lambda = 0$.

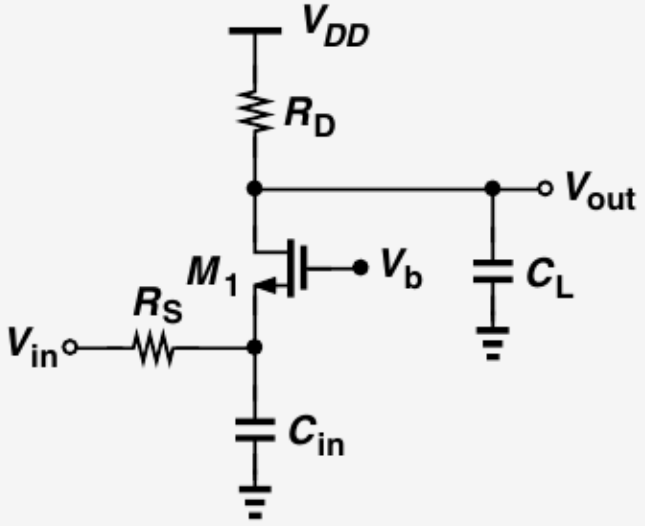


Figure 11.11