Chap. 9

The Complete Response of Circuits with Two Energy Storage Elements (2)

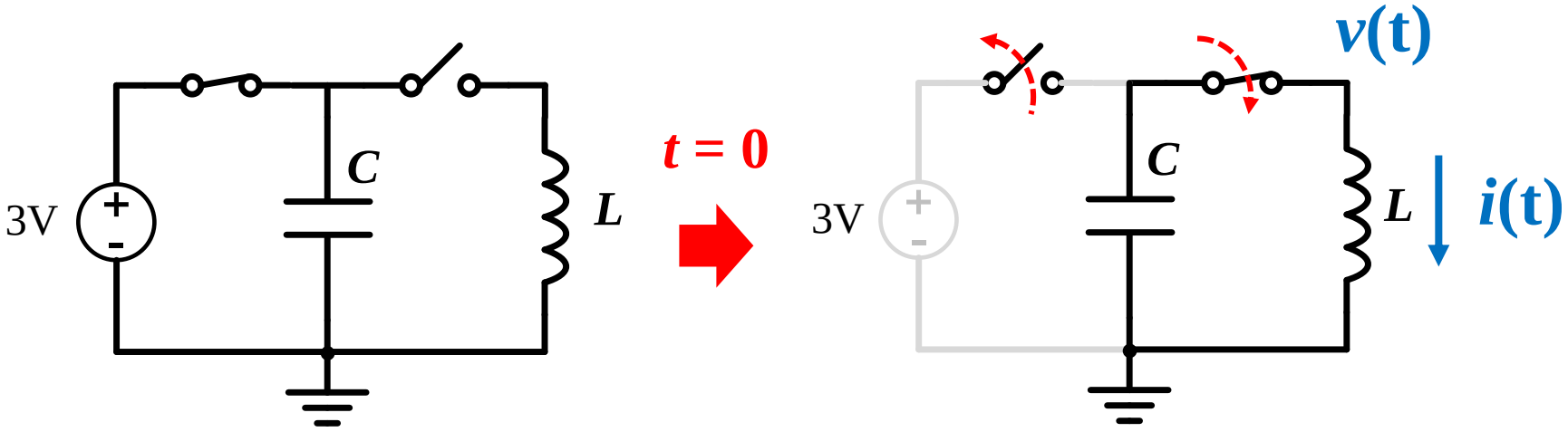
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Natural Response of RLC Circuits

- ✓ 9.4 Natural Response of the Unforced Parallel RLC Circuit
- ✓ 9.5 Natural Response of the Critically Damped Unforced Parallel RLC Circuit
- ✓ 9.6 Natural Response of an Underdamped Unforced Parallel RLC Circuit

Oscillation in LC Circuit



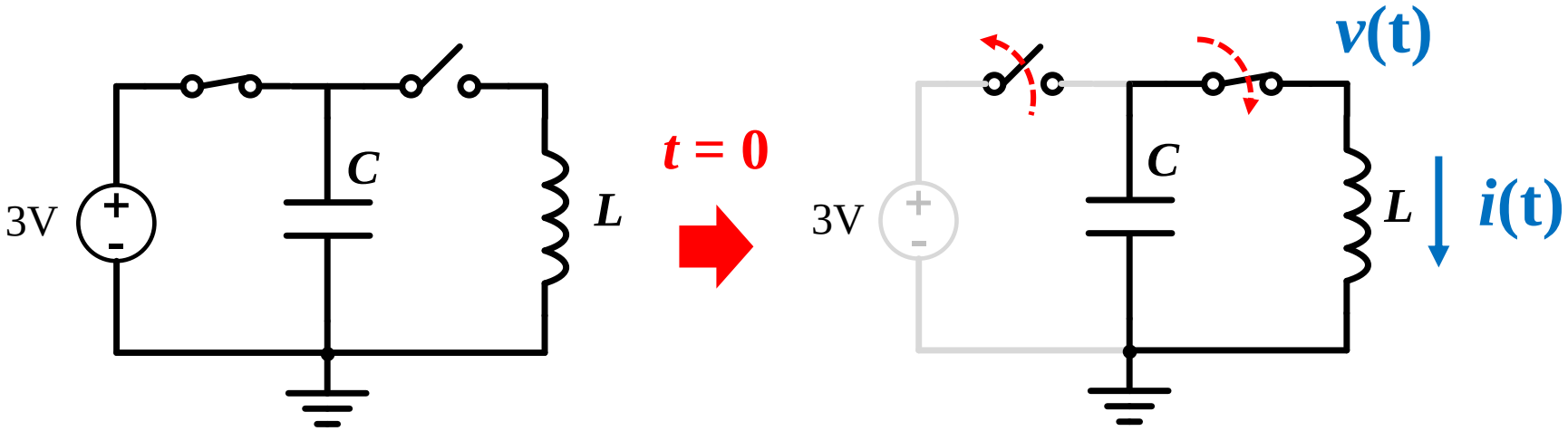
Inductor : $v(t) = L\frac{di(t)}{dt}$ **Capacitor:** $-i(t) = C\frac{dv(t)}{dt}$

$$v(t) = -LC\frac{d^2v(t)}{dt^2}$$

$$v(t) + LC\frac{d^2v(t)}{dt^2} = 0$$

$$v(t) = A\exp(+j\frac{1}{\sqrt{LC}}t) + B\exp(-j\frac{1}{\sqrt{LC}}t)$$

Oscillation in LC Circuit (cont'd)



$$v(t) = A\exp(+j\frac{1}{\sqrt{LC}}t) + B\exp(-j\frac{1}{\sqrt{LC}}t)$$

$$v(t) = A\exp(+j\frac{1}{\sqrt{LC}}t) + (3-A)\exp(-j\frac{1}{\sqrt{LC}}t)$$

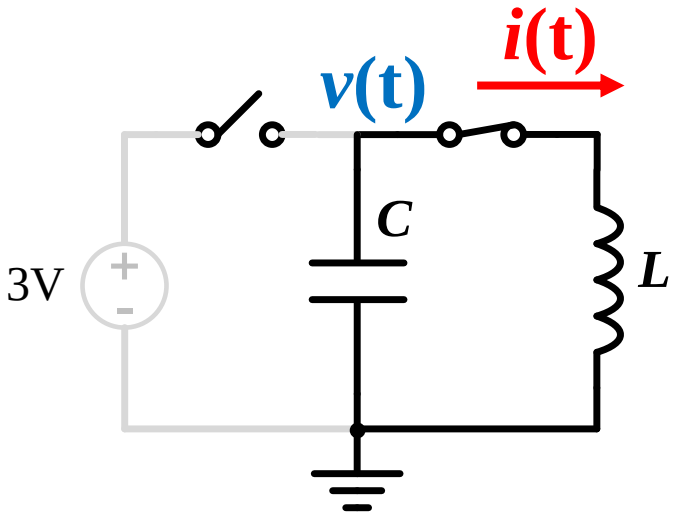
$$v(t) = \frac{3}{2}\left\{\exp(+j\frac{1}{\sqrt{LC}}t) + \exp(-j\frac{1}{\sqrt{LC}}t)\right\}$$

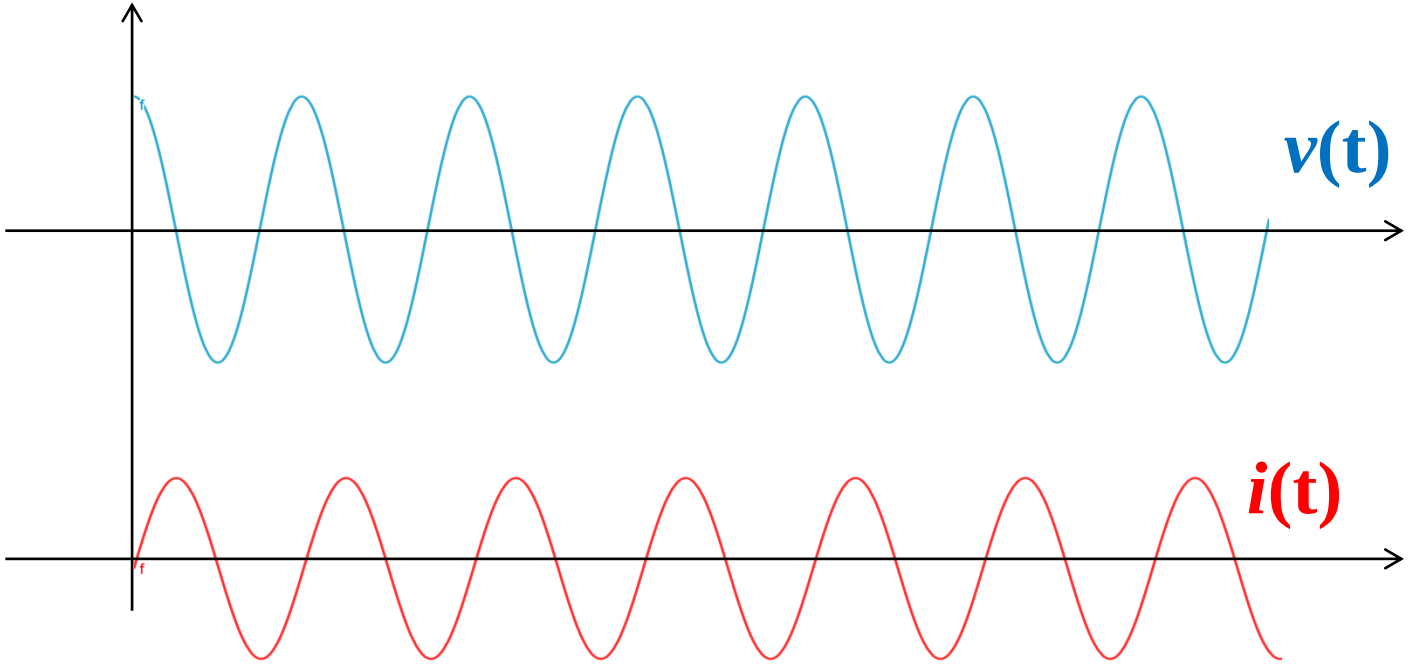
$$v(t) = 3\cos\frac{1}{\sqrt{LC}}t$$

Waveform of LC Circuit

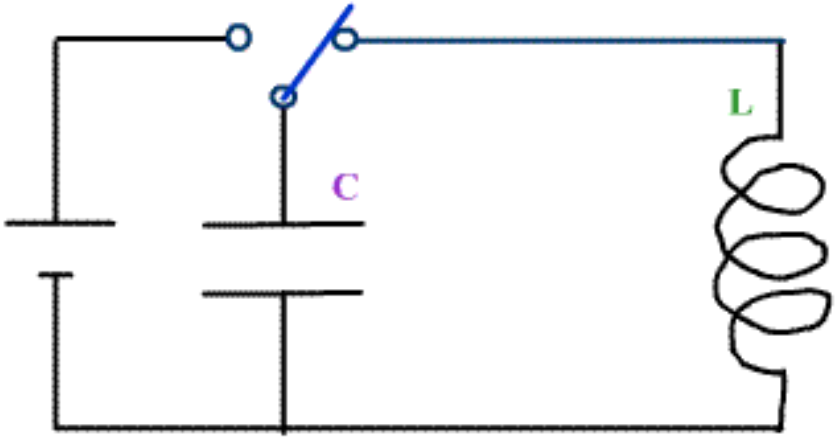
$$v(t) = 3\cos\frac{1}{\sqrt{LC}}t$$

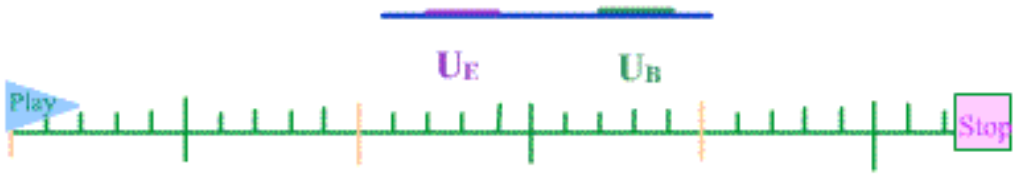
$$i(t) = 3\sqrt{\frac{C}{L}}\sin\frac{1}{\sqrt{LC}}t$$



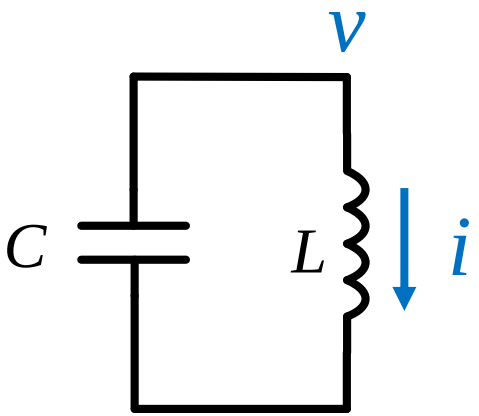


LC Oscillation?

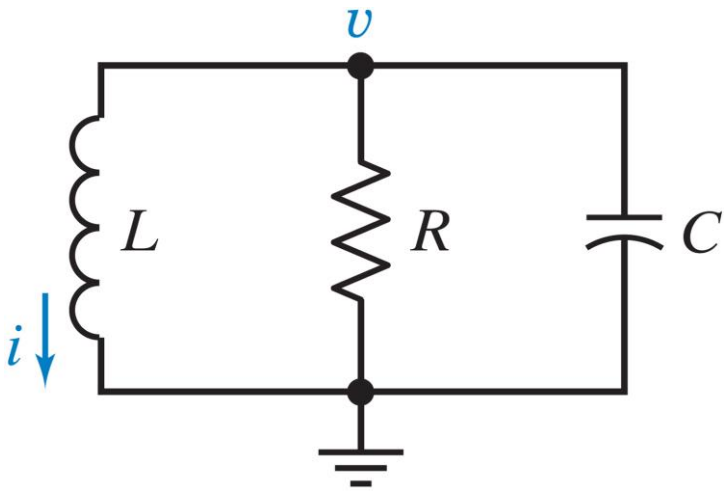




LC vs. RLC

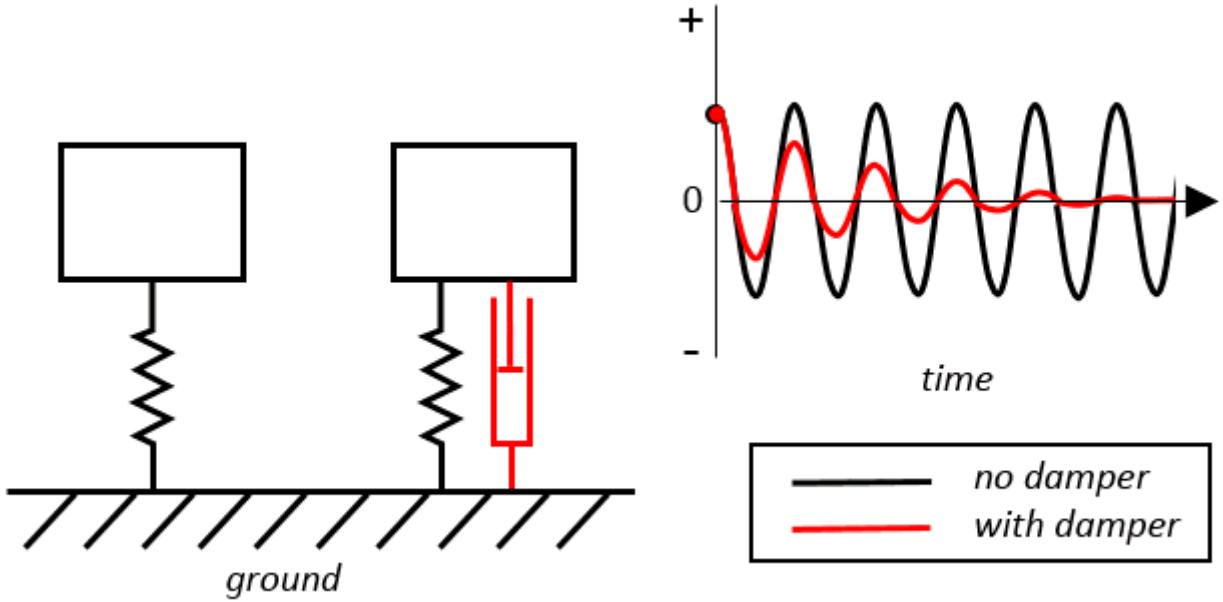


Oscillation

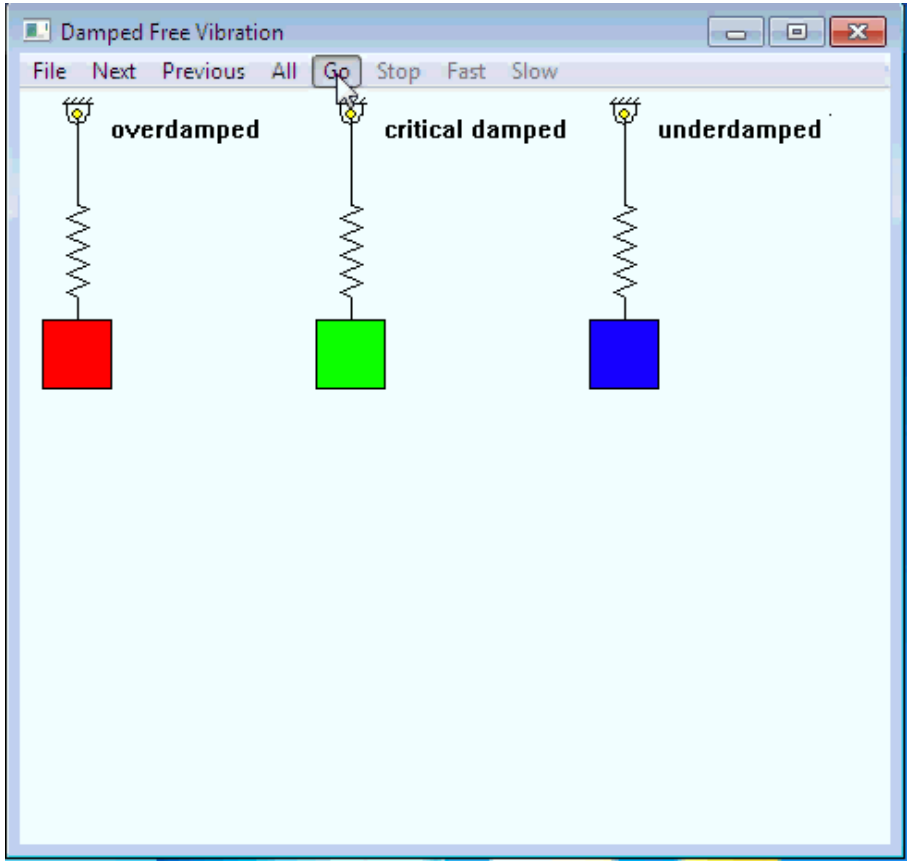


Oscillation & Damping

What is Damping?



Degree of Damping



General Form of Differential Equation RLC Circuit w/ One L & One R

- A RLC circuit containing 1 inductor and 1 capacitor is represented as

f(t) : Input

$$\frac{d^2}{dt^2}x(t) + b\frac{d}{dt}x(t) + \mathrm{c}\,x(t) = f(t)$$

x(t) : Output

- Rewriting the above equation in the respect of the frequency response, (allocating $2\alpha = b$, $\omega_0^2 = c$)

$$\frac{d^2}{dt^2}x(t) + 2\alpha\frac{d}{dt}x(t) + \omega_0^2 x(t) = f(t)$$

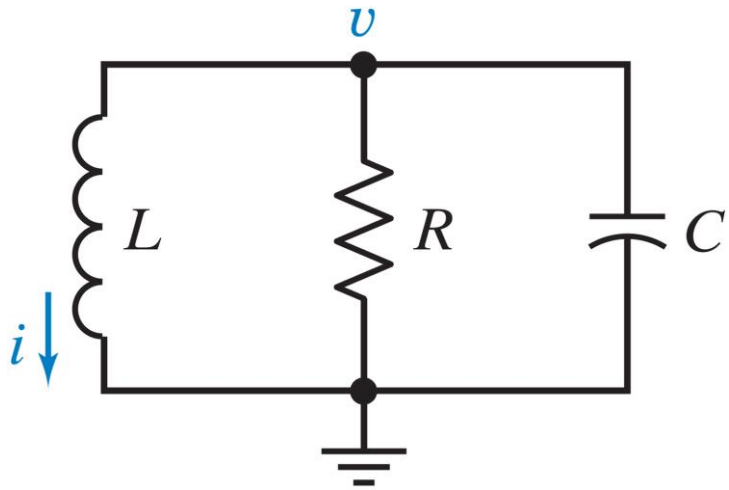
where

α = **damping coefficient**,

ω_0 = **resonant frequency**

Diff. Eq of Parallel RLC circuit

- Define v & i as follows
  - v : Voltage applied for C
  - i : Current flowing through L



- Applying KCL,

$$\frac{v}{R} + i + C\frac{dv}{dt} = 0$$

- Differentiating with t,

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = 0$$

Characteristic Eq. of Parallel RLC

- Dividing both sides by C and using operator s,

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$: Characteristic Equation

- Two roots of characteristic equation

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \quad \text{and} \quad s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

- When $s_1 \neq s_2$, the solution of diff eq. or, **Natural response**, is

$$v_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Re-Writing Diff. Eq.

- Revisiting the previous Diff. Eq,

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=0$$

- The above equation can be seen as

$$\frac{d^2v}{dt^2}+2\alpha\frac{dv}{dt}+\omega_0^2=0 \quad \text{with} \quad \alpha=\frac{1}{2RC},\ \omega_0^2=\frac{1}{LC}$$

Two roots of the characteristic equation are rewritten as

$$s_1=-\alpha+\sqrt{\alpha^2-\omega_0^2} \quad \text{and} \quad s_2=-\alpha-\sqrt{\alpha^2-\omega_0^2}$$

where the **damped** resonant frequency is defined as

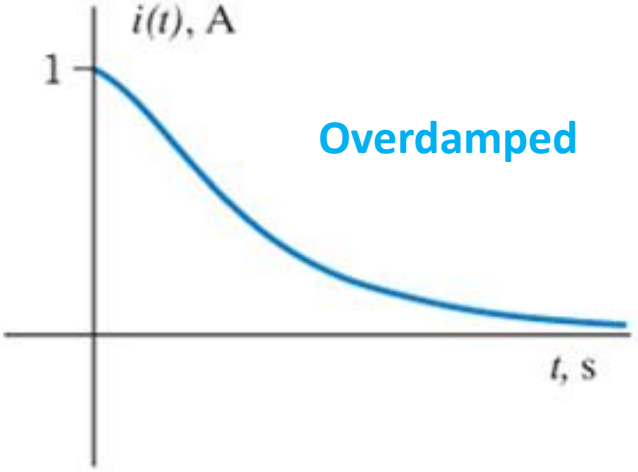
$$\omega_d=\sqrt{\omega_0^2-\alpha^2}$$

: Meaningful when the roots becomes complex number.

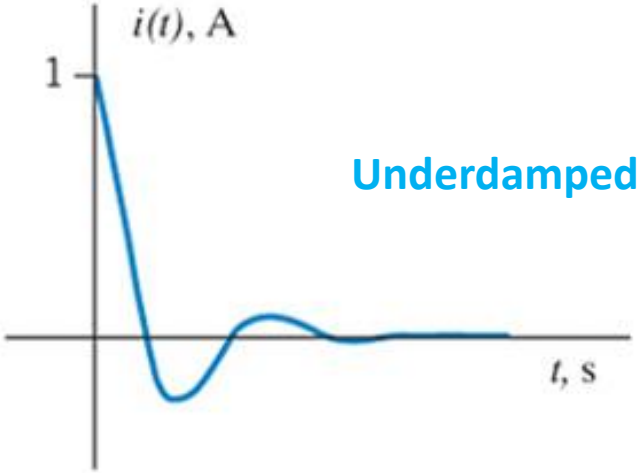
Four Possible Conditions of The Roots of Characteristic Equation

- Two real and distinct roots when $\alpha^2 > \omega_0^2$ **: Overdamped**
- Two real equal roots when $\alpha^2 = \omega_0^2$ **: Critically Damped**
- Two complex roots when $\alpha^2 < \omega_0^2$ **: Underdamped**
- Two pure imaginary roots when $\alpha = 0$ **: Undamped**

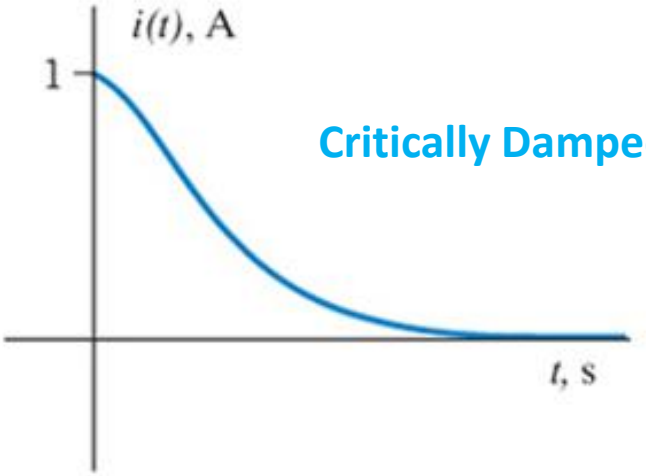
Natural Response of a Parallel RLC Circuit



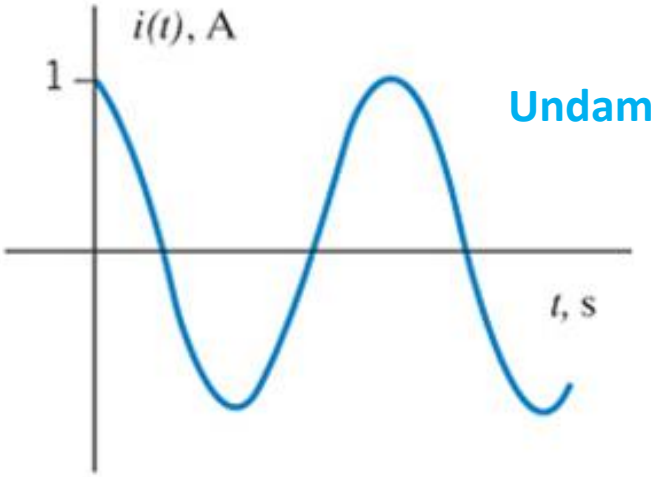
Overdamped



Underdamped



Critically Damped

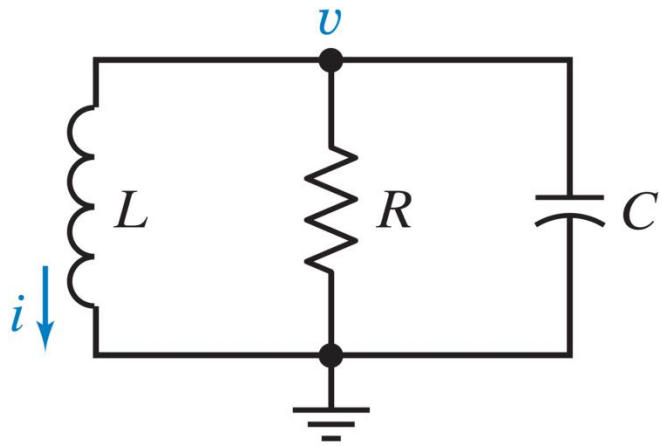


Undamped

Ex 9.4-1 : Overdamped 2nd Order Circuit

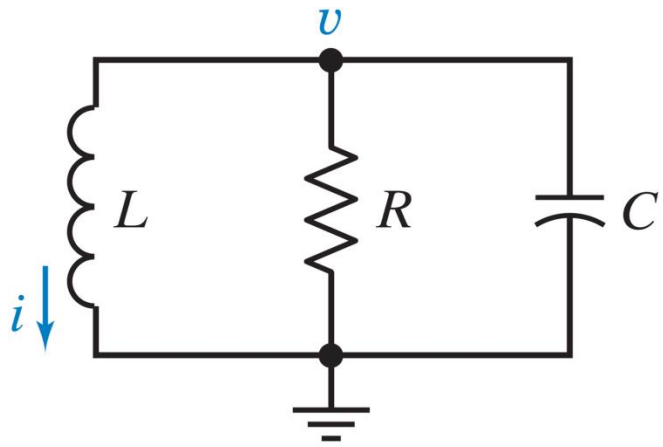
- Find the natural response of v(t) for t>0 when R = 2/3 Ω, L=1H, C = 1/2F, and v(0)= 10V, and i(0)=2A.

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$



EXERCISE 9.4-1

- Repeat it for R = 6 Ω, L = 7H, C = 1/42F, and v(0)= 0V, and i(0) = 10A.



Critically Damped: When $s_1 = s_2$

- Revisiting parallel RLC characteristic diff. equation,

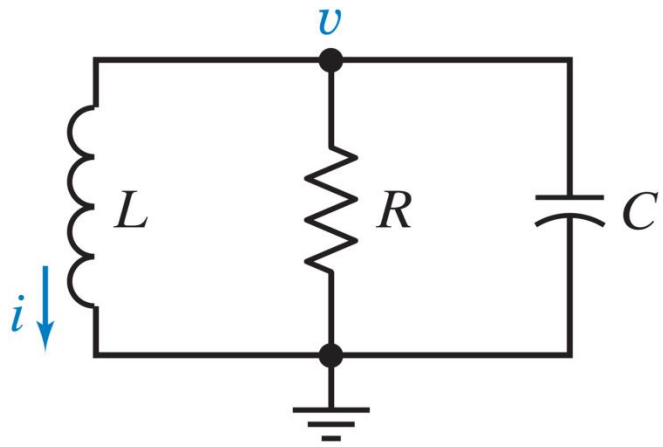
$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \qquad \alpha = \frac{1}{2RC},\ \omega_0^2 = \frac{1}{LC}$$

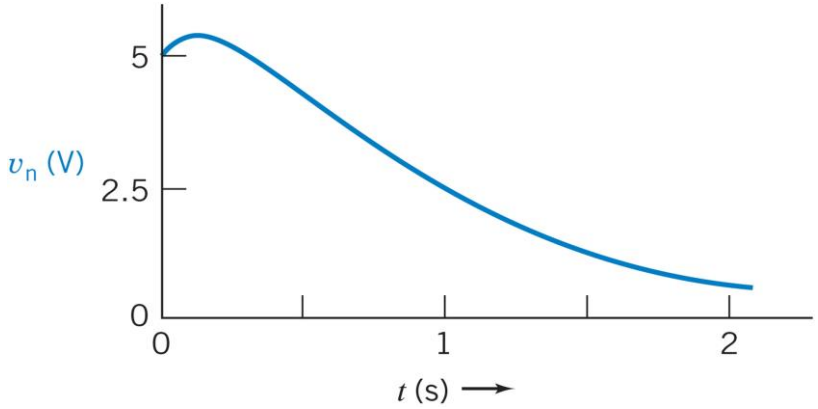
- If $s_1=s_2$, or $\alpha^2 = \omega_0^2$, $v_n = A \exp(s_1 t)$?
  ➔ No it is not possible, we have two initial conditions.
- In this case, we try the solution

$$v_n = e^{s_1 t}(A_1 t + A_2)$$

Example for Critically Damped

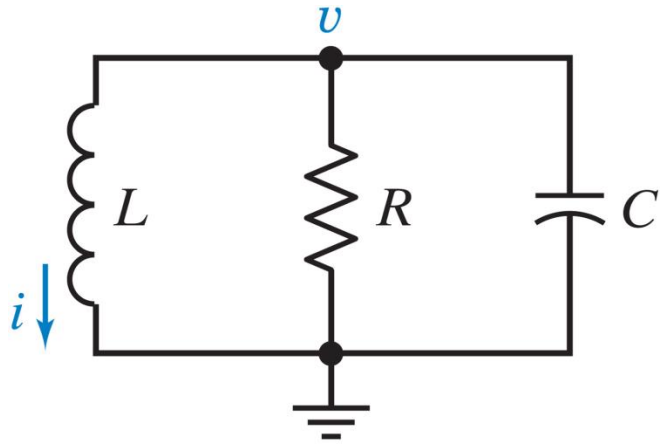
- L=1H, R=1Ω, C=1/4F with v(0) = 5V, i(0)=-6A





EXERCISE 9.5-1

- L=0.4H, R=10Ω, C=1mF with v(0) = 8V, i(0)= 0.



Underdamped : Two Complex Roots

- Same as the overdamped case,

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

$$v_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \quad \text{and} \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \quad \text{with} \quad \alpha = \frac{1}{2RC},\ \omega_0^2 = \frac{1}{LC}$$

- As $\alpha^2 < \omega_0^2$, two roots can be represented as

$$s_{1,2} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha_0^2} = -\alpha \pm j\omega_d \quad \text{with} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

Damped resonant frequency

$$v_n = e^{-\alpha t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right)$$

Natural Response of Underdamped 2nd Order Circuit

- Utilizing Euler identity,

$$e^{\pm j\omega t} = \cos \omega t \pm j \sin \omega t$$

- Natural response can be re-written as

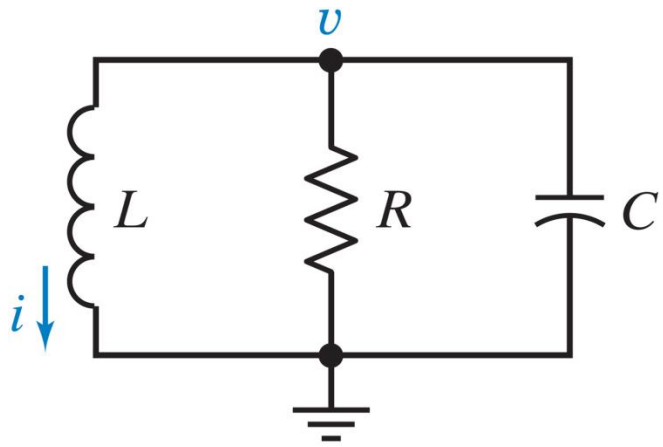
$$v_n = e^{-\alpha t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right)$$

$$= e^{-\alpha t}\left[(A_1 + A_2)\cos \omega_d t + j(A_1 - A_2)\sin \omega_d t\right]$$

$$v_n = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

Example 9.6-1

- L=0.1H, R=25/3Ω, C=1mF with v(0) = 10V, i(0)= -0.6A.



EXERCISE 9.6-1

- L=10mH, R = 62.5Ω, C=1μF with v(0) = 10V, i(0)= 80mA.

