Chap. 13

Frequency Response

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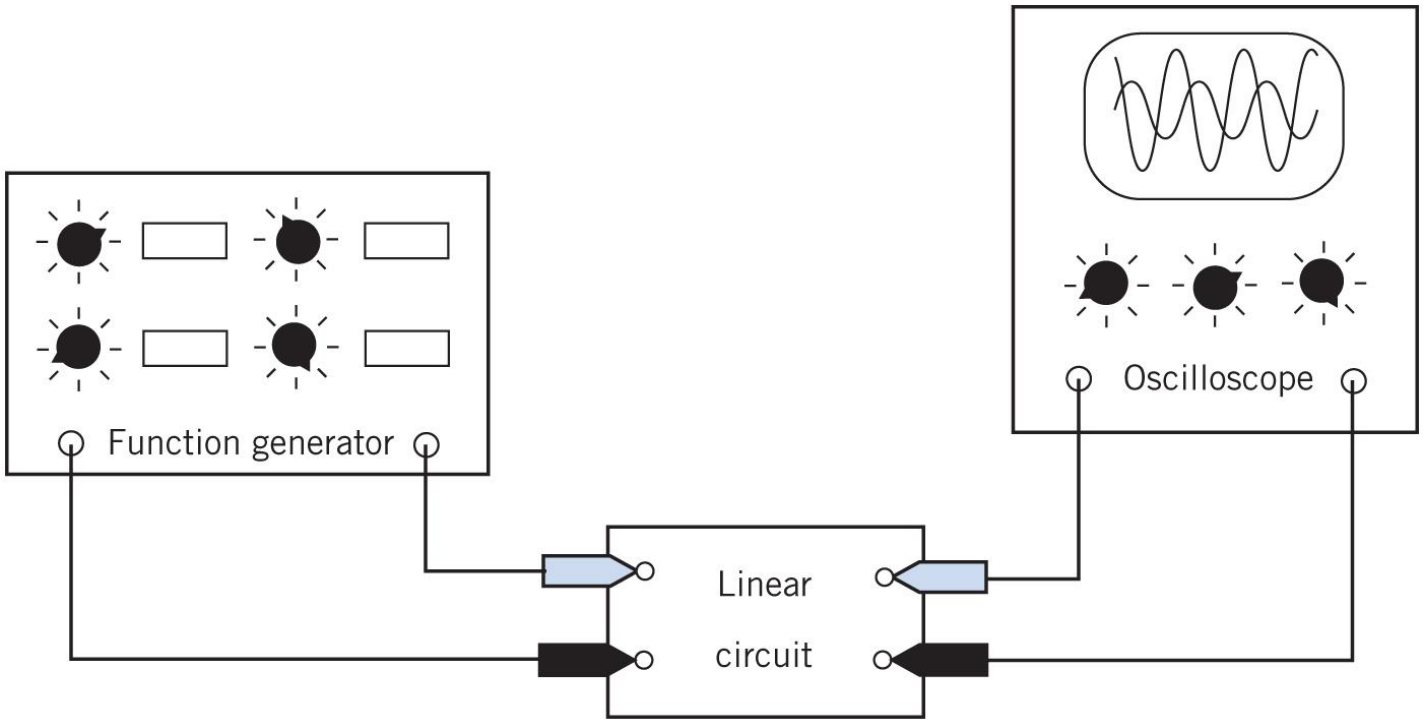
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Frequency Response & Bode Plot

- ✓ 13.2 Gain, Phase Shift, and the Network Function
- ✓ 13.3 Bode Plots

Introduction

- At any fixed frequency,
  - Output amplitude / Input amplitude = constant
  - Output phase – Input phase = constant
- How about for varying frequency?



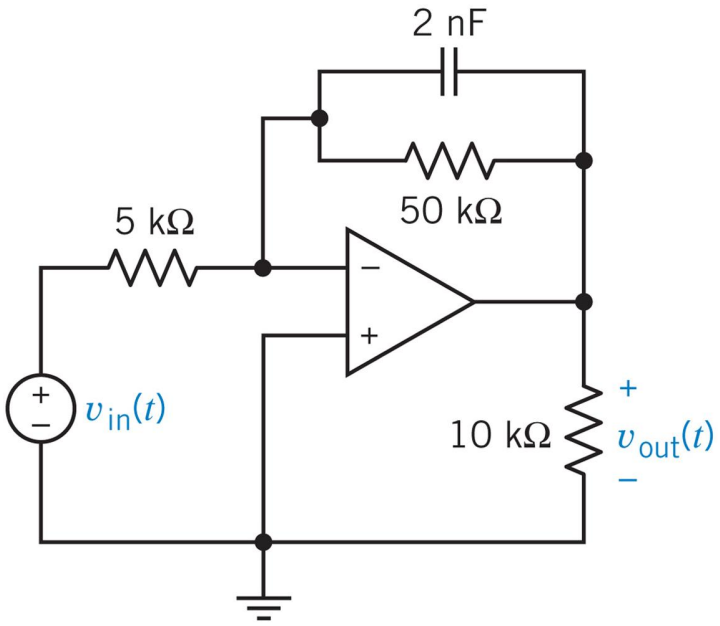
Gain & Phase Shift

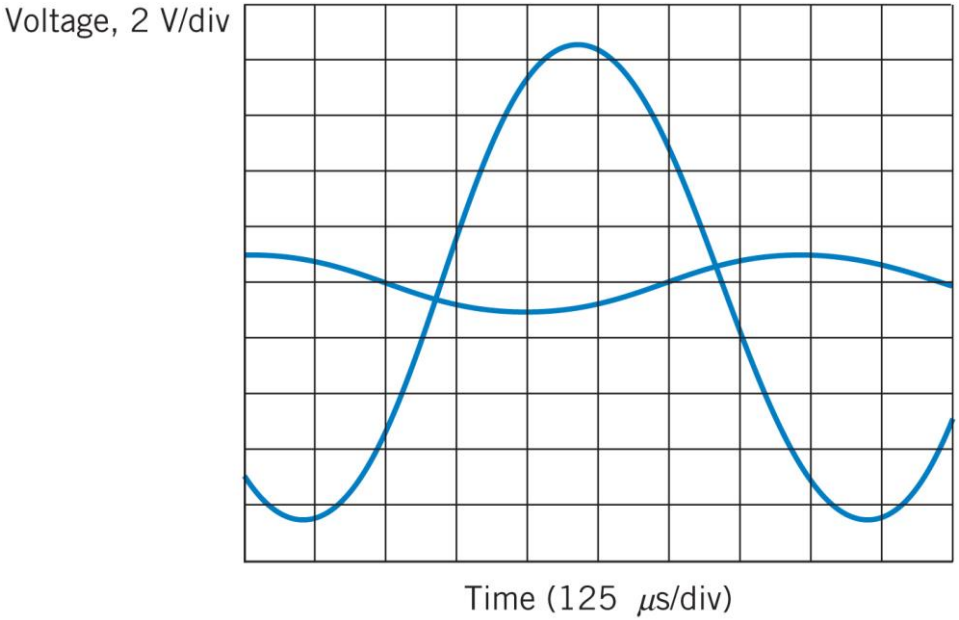
- Sinusoidal input → Sinusoidal output

$$v_{\text{in}}(t) = A\cos\omega t \quad \rightarrow \quad v_{\text{out}}(t) = B\cos(\omega t + \theta)$$

- Then,

$$\text{gain} = \frac{B}{A} \qquad \text{phase shift} = \theta - 0^\circ = \theta$$





Phase Lead & Phase Lag

- Suppose that

$$v_{\text{in}}(t) = 1 \cos 6283t \text{ V} \qquad v_{\text{out}}(t) = 8.47 \cos (6283t + 148^\circ) \text{ V}$$

- You can represent it as time shift as

$$B \cos (\omega t + \theta) = B \cos \left(\omega \left(t + \frac{\theta}{\omega} \right) \right) = B \cos (\omega (t + t_0))$$

- For the given case,

$$v_{\text{out}}(t) = 8.47 \cos (6283t + 148^\circ) = 8.47 \cos (6283t - 212^\circ)$$

- Which means,

$$t_0 = \frac{\theta}{\omega} = \frac{2.58 \text{ rad}}{6283 \text{ rad/s}} = 410 \ \mu\text{s} \qquad t_0 = \frac{-3.70 \text{ rad}}{6283 \text{ rad/s}} = -590 \ \mu\text{s}$$

➔ At ω = 6283 rad/s, gain = 8.47 and phase **lead** of 148° or phase **lag** of 212°. Or, the circuit advances input by 410 µs or delays is by 590 µs.

Changing Frequency

- The gain and phase shift changes when input frequency is changed.

Table 13.2-1 Frequency Response Data for a Circuit

f (Hz)	ω (rad/s)	GAIN	PHASE SHIFT
100	628.3	9.98	176°
500	3,141.6	9.54	163°
1,000	6,283	8.47	148°
5,000	31,416	3.03	108°
10,000	62,830	1.57	99°

- Can we express this as function?

Network Function

- X(ω) : phasor for input, Y(ω) : phasor for steady state response
- Network function :

$$\mathbf{H}(\omega) = \frac{\mathbf{Y}(\omega)}{\mathbf{X}(\omega)}$$

- Here,

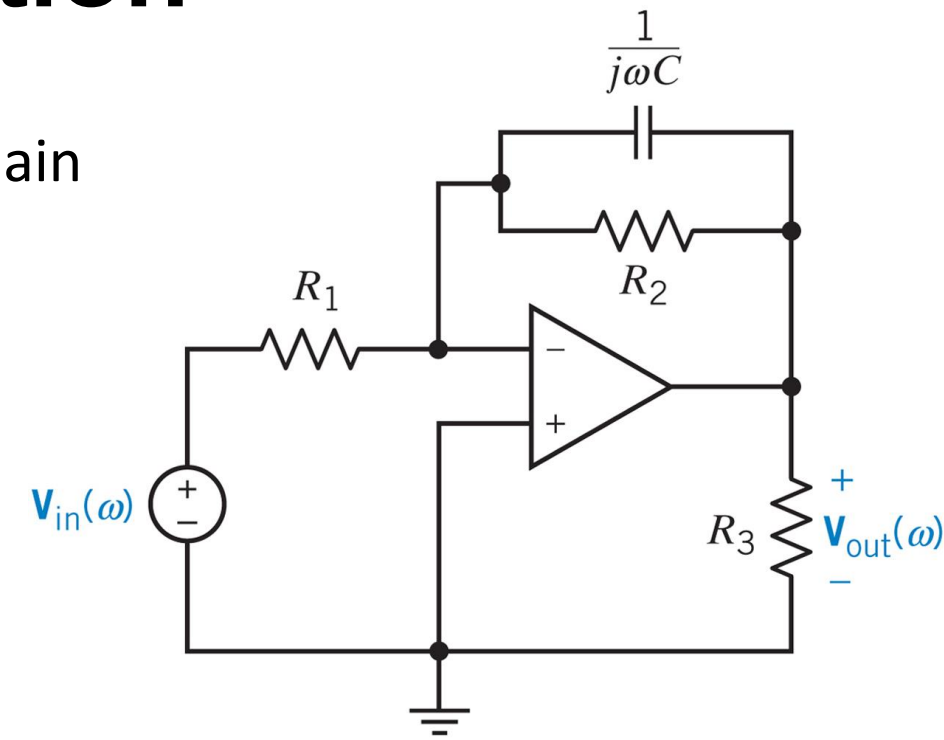
$$\text{gain} = |\mathbf{H}(\omega)| = \frac{|\mathbf{Y}(\omega)|}{|\mathbf{X}(\omega)|}$$

$$\text{phase shift} = \angle\mathbf{H}(\omega) = \angle\mathbf{Y}(\omega) - \angle\mathbf{X}(\omega)$$

- How can we find it?

Finding Network Function

- Represent the circuit with frequency domain
- Using KCL & KVL to find out the relation between $V_{in}(\omega)$ and $V_{out}(\omega)$.



$$\frac{\mathbf{V}_{\text{in}}(\omega)}{R_1} + \frac{\mathbf{V}_{\text{out}}(\omega)}{R_2} + j\omega C\mathbf{V}_{\text{out}}(\omega) = 0$$

- This implies

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_{\text{out}}(\omega)}{\mathbf{V}_{\text{in}}(\omega)} = \frac{-R_2}{R_1 + j\omega C R_1 R_2}$$

- Gain & Phase shift

$$\text{gain} = |\mathbf{H}(\omega)| = \text{H} = \frac{R_2/R_1}{\sqrt{1+\omega^2 C^2 R_2^2}} \quad \text{phase shift} = \underline{/\mathbf{H}(\omega)} = 180^\circ - \tan^{-1}(\omega C R_2)$$

Verification of H(ω)

- Derived H(ω) :

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_{\text{out}}(\omega)}{\mathbf{V}_{\text{in}}(\omega)} = \frac{-R_2}{R_1 + j\omega C R_1 R_2}$$

- When R1 = 5kΩ, R2 = 50kΩ, and C = 2nF,

$$\mathbf{H}(\omega) = \frac{-10}{1 + (j\omega/10,000)}$$

$$\text{gain} = |\mathbf{H}(\omega)| = \frac{10}{\sqrt{1 + (\omega^2/10^8)}}$$

$$\text{phase shift} = \angle\mathbf{H}(\omega) = 180^\circ - \tan^{-1}(\omega/10,000)$$

- Inserting ω = 6238 rad/s,

$$\text{gain} = \frac{10}{\sqrt{1 + \frac{6283^2}{10^8}}} = 8.47$$

$$\text{phase shift} = 180^\circ - \tan^{-1}(6283/10,000) = 148^\circ$$

Frequency Response

- What does we say when “frequency response”?

➔Gain & Phase shift as function of frequency!

- For example of ω = 5000 rad/s,

$$\mathbf{H}(\omega) = \frac{-10}{1 + (j5000/10,000)} = 8.94 \angle 153^\circ$$

- Next, we can derive V_{out} from V_{in},

$$v_{\text{in}}(t) = 0.4 \cos(5000t + 45^\circ)\ \text{V}$$

$$\mathbf{V}_{\text{out}}(\omega) = \mathbf{H}(\omega)\mathbf{V}_{\text{in}}(\omega) = (8.94 \angle 153^\circ)(0.4 \angle 45^\circ) = 3.58 \angle 198^\circ$$

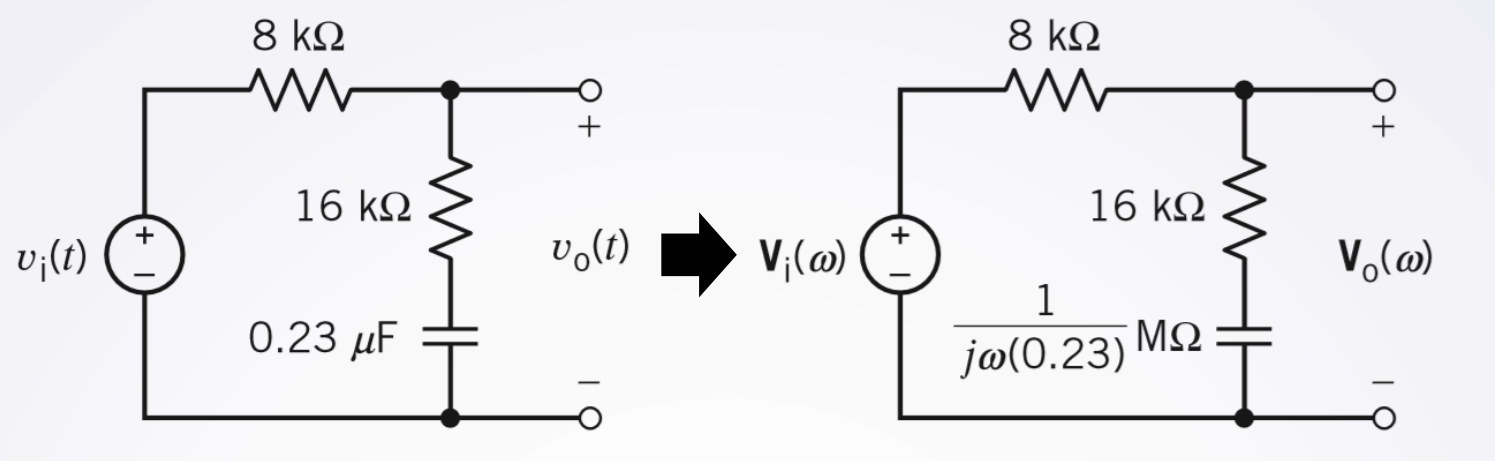
- In time domain,

$$v_{\text{out}}(t) = 3.58 \cos(5000t + 198^\circ)\ \text{V}$$

Example 13.2-1

- The network function that represent this circuit has the form:

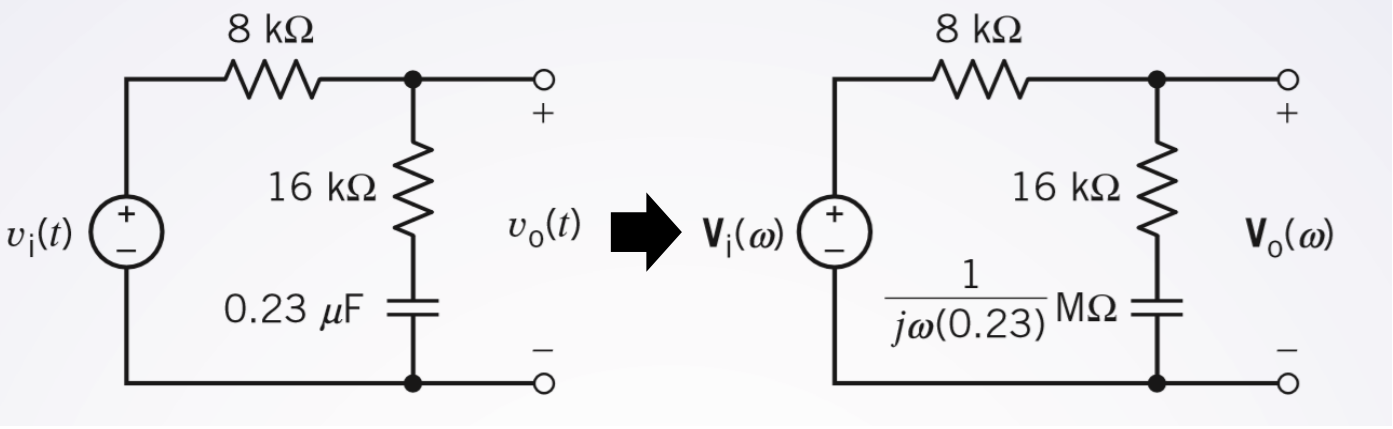
$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} = \frac{1 + j\frac{\omega}{z}}{1 + j\frac{\omega}{p}}$$



- The parameter z is called the zero of the circuit and the parameter p is called the pole of the circuit. Determine the values of z and of p for the circuit in Figure 13.2-4a.

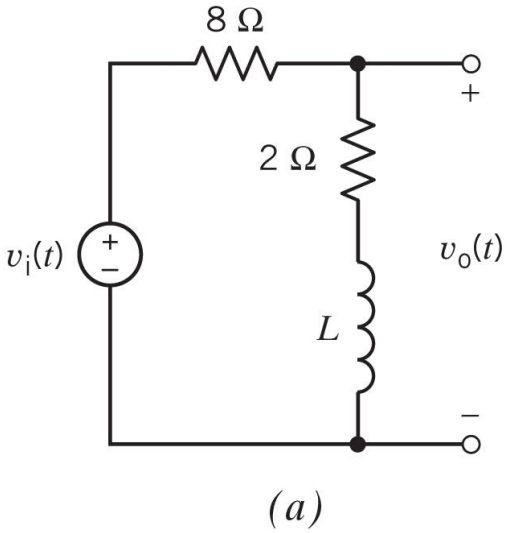
Example 13.2-1

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} = \frac{1 + j\dfrac{\omega}{z}}{1 + j\dfrac{\omega}{p}}$$

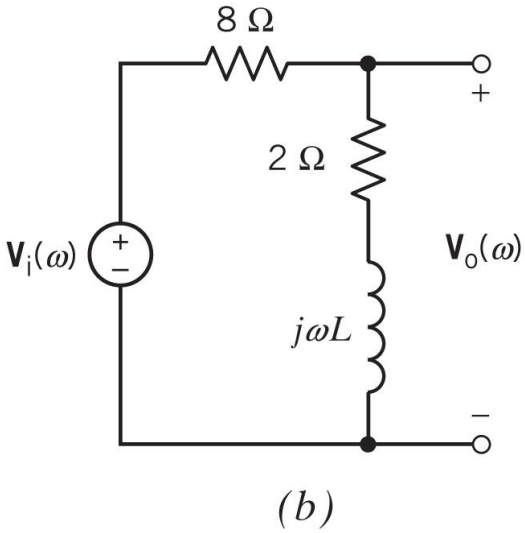


Example 13.2-2

- Determine the value of the inductance L when the network function that represent circuit is



(a)

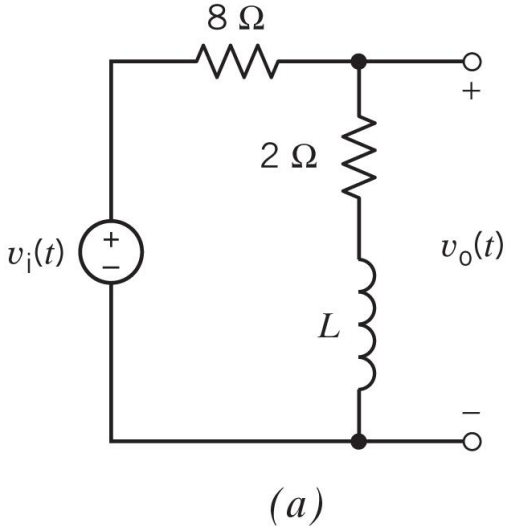


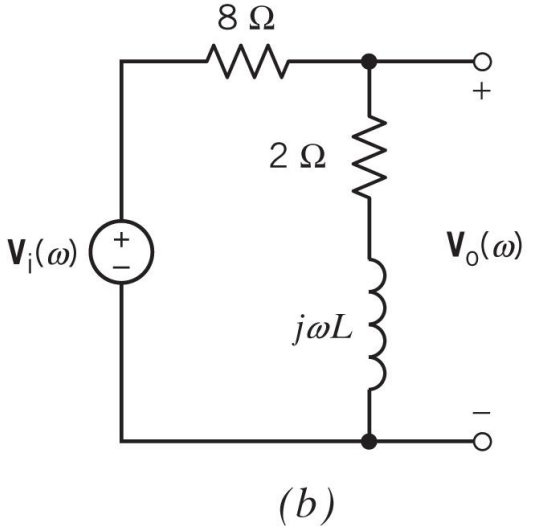
(b)

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} = 0.2\frac{1 + j\frac{\omega}{5}}{1 + j\frac{\omega}{25}}$$

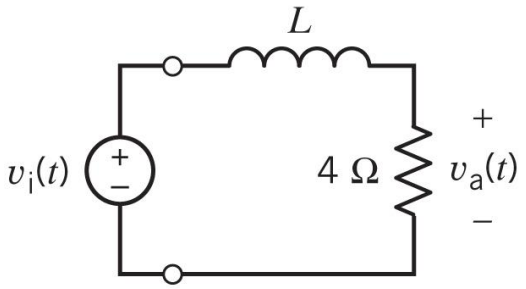
Example 13.2-2

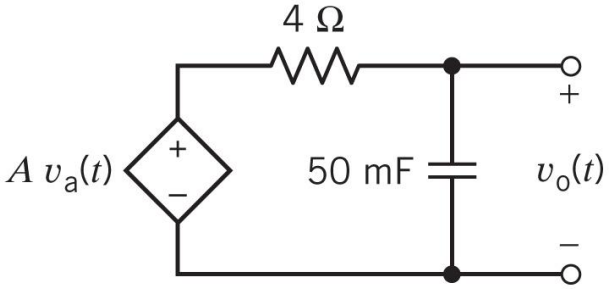
$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} = 0.2\frac{1 + j\dfrac{\omega}{5}}{1 + j\dfrac{\omega}{25}}$$





Example 13.2-3

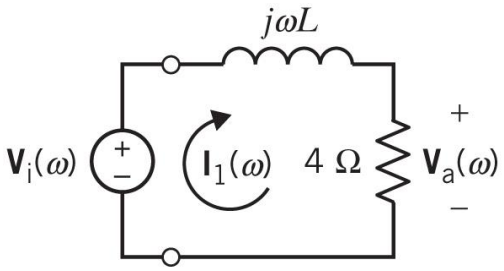


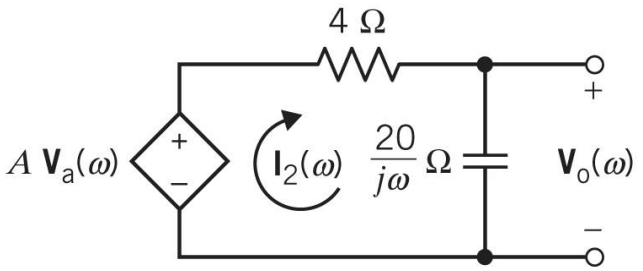


- Determine the value of the inductance L and the gain A of the of the voltage-controlled voltage source, when

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} = \frac{3}{\left(1 + j\frac{\omega}{2}\right)\left(1 + j\frac{\omega}{5}\right)}$$

Example 13.2-3

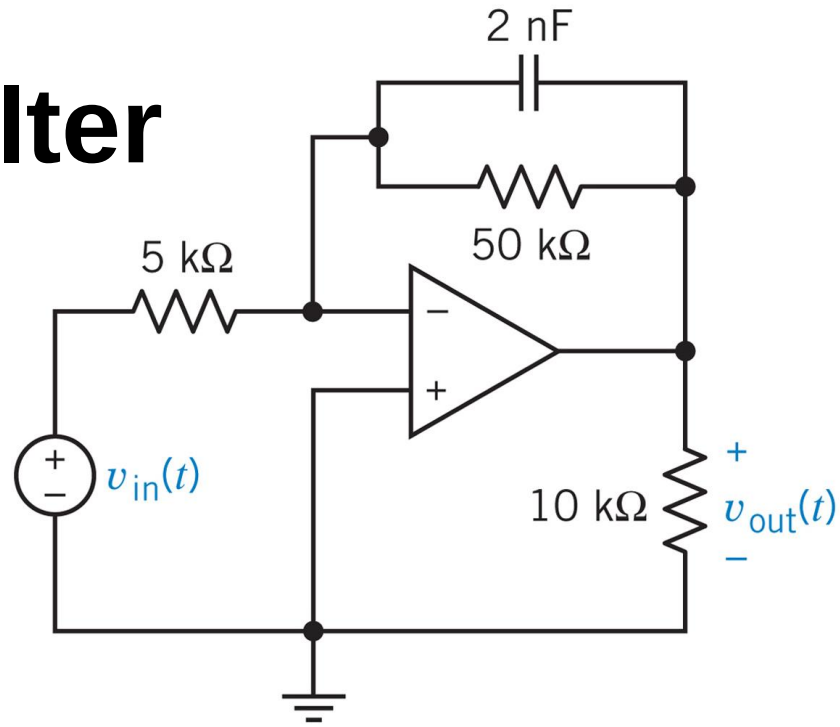




$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} = \frac{3}{\left(1 + j\frac{\omega}{2}\right)\left(1 + j\frac{\omega}{5}\right)}$$

Revisit RC Circuit & LC Circuit

First-order Low-Pass Filter



- The circuit shown in the figure has the network function of the form:

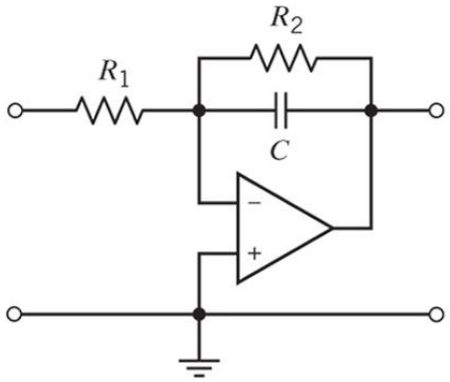
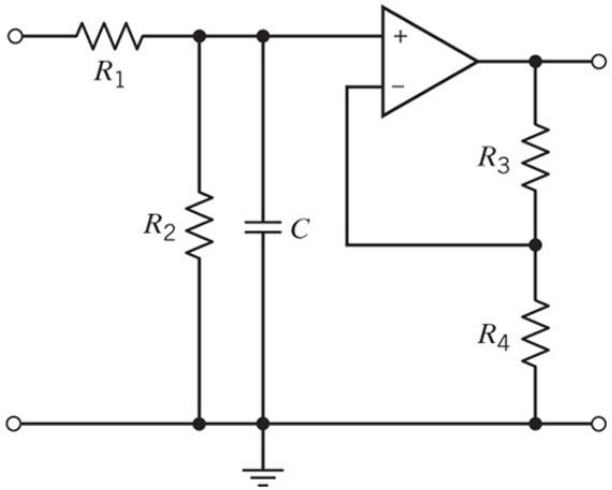
$$\mathbf{H}(\omega) = \frac{H_0}{1 + j\dfrac{\omega}{\omega_0}}$$

- The circuit whose network function form above is called "**first order low pass filter**". Why?
- Gain and phase shift of the first order low pass filter are

$$\text{gain} = \frac{|H_0|}{\sqrt{1 + \dfrac{\omega^2}{\omega_0^2}}} \qquad \text{phase shift} = \angle H_0 - \tan^{-1}(\omega/\omega_0)$$

- At low frequencies – $\omega << \omega_0$ – the gain is $|H_0|$, which is called **dc gain** and ω_0 is called half-power frequency.

Two Types of LP-Filter

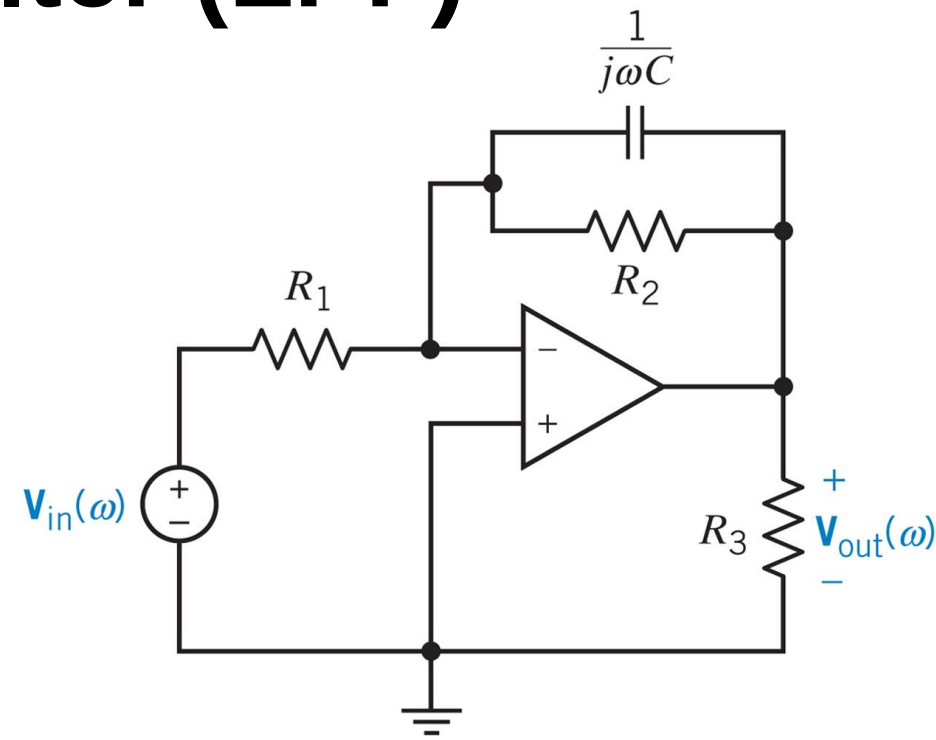
PHASE SHIFT	FIRST-ORDER LOW-PASS FILTER CIRCUIT	DESIGN EQUATIONS
$90° \leq \text{phase shift} \leq 180°$		$H_0 = -\frac{R_2}{R_1}$ $\omega_0 = \frac{1}{R_2 C}$
$-90° \leq \text{phase shift} \leq 0°$		$H_0 = \frac{R_2}{R_1 + R_2}\left(1 + \frac{R_3}{R_4}\right)$ $\omega_0 = \frac{R_1 + R_2}{R_1 R_2 C}$

Design of Low Pass Filter (LPF)

- Can you design low-pass filter (LPF) that satisfies the following specification?

dc gain = 2

phase shift = 120° when ω = 1000 rad/s



Bode Plot

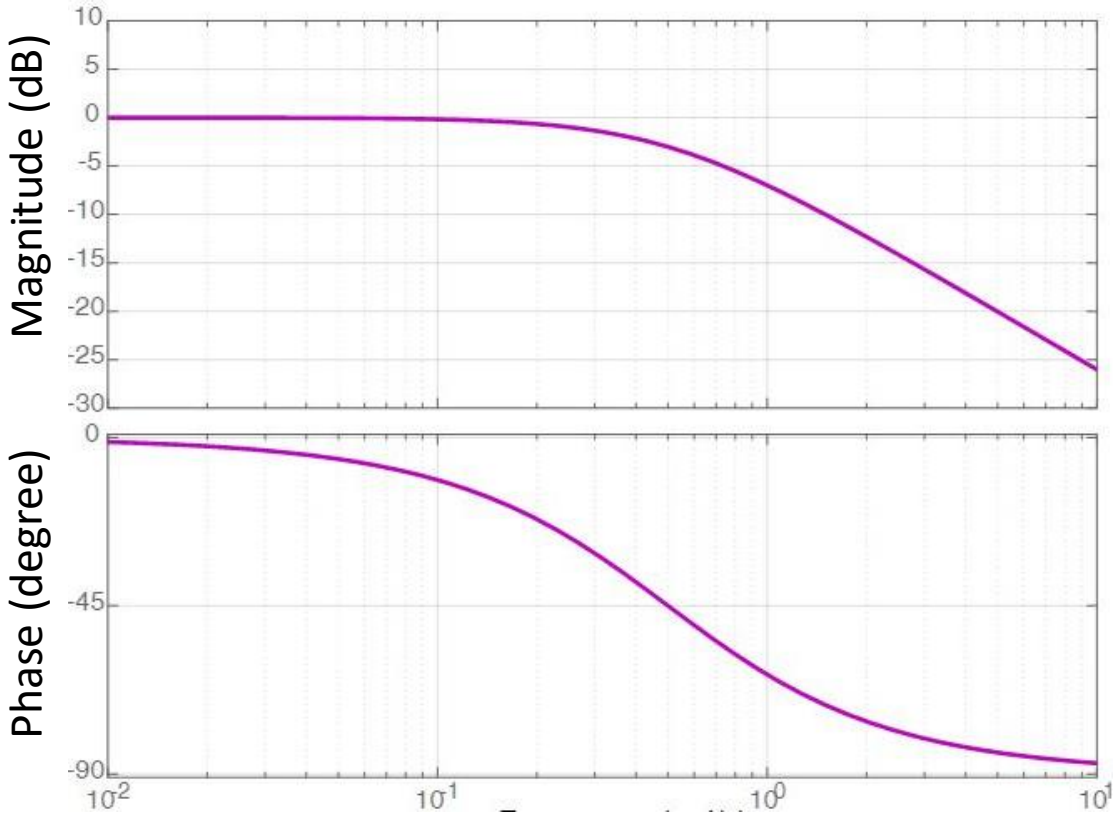
- The Bode plot is a chart of **gain in decibels and phase in degrees** versus the **logarithm of frequency**.
- The network function **H** can be written as:

$$\mathbf{H} = \mathrm{H} \angle \phi = \mathrm{H}e^{j\phi}$$

- The logarithm of the magnitude, the gain in dB:

$$\text{logarithmic gain} = 20 \log_{10} \mathrm{H}$$

Unit : decibel (dB)



Bode Plot of 1st order LPF

- Example of the network function:

$$\mathbf{H} = \frac{1}{1 + j\dfrac{\omega}{\omega_0}} = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}} \angle \tan^{-1}(\omega/\omega_0) = H \angle \phi$$

- The logarithm gain is

$$20 \log_{10} H = 20 \log_{10} \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}$$

$$= 20 \log_{10} 1 - 20 \log_{10} \sqrt{1 + (\omega/\omega_0)^2} = -20 \log_{10} \sqrt{1 + (\omega/\omega_0)^2}$$

- Bode plot should be

Gain : **$20 \log_{10} H$ vs. $\log_{10} \omega$**

Phase : **ϕ vs. $\log_{10} \omega$**

- Please note that horizontal axis is log scale, not linear scale.

Can You Draw It by Hand?

Approximation Strategy of 1st order LPF Magnitude Bode Plot

- Can we draw the below gain in dB? We can approximate it!

$$20\log_{10} H = 20\log_{10}\frac{1}{\sqrt{1+(\omega/\omega_0)^2}}$$

$$= 20\log_{10} 1 - 20\log_{10}\sqrt{1+(\omega/\omega_0)^2} = -20\log_{10}\sqrt{1+(\omega/\omega_0)^2}$$

- For small frequencies, $\omega << \omega_0$, $1+(\omega/\omega_0)^2 \cong 1$

$$20\log_{10} H = -20\log_{10}\sqrt{1} = 0\ \text{dB}$$

- For large frequencies, $\omega >> \omega_0$, $1+(\omega/\omega_0)^2 \cong (\omega/\omega_0)^2$

$$20\log_{10} H = -20\log_{10}\sqrt{(\omega/\omega_0)^2}$$

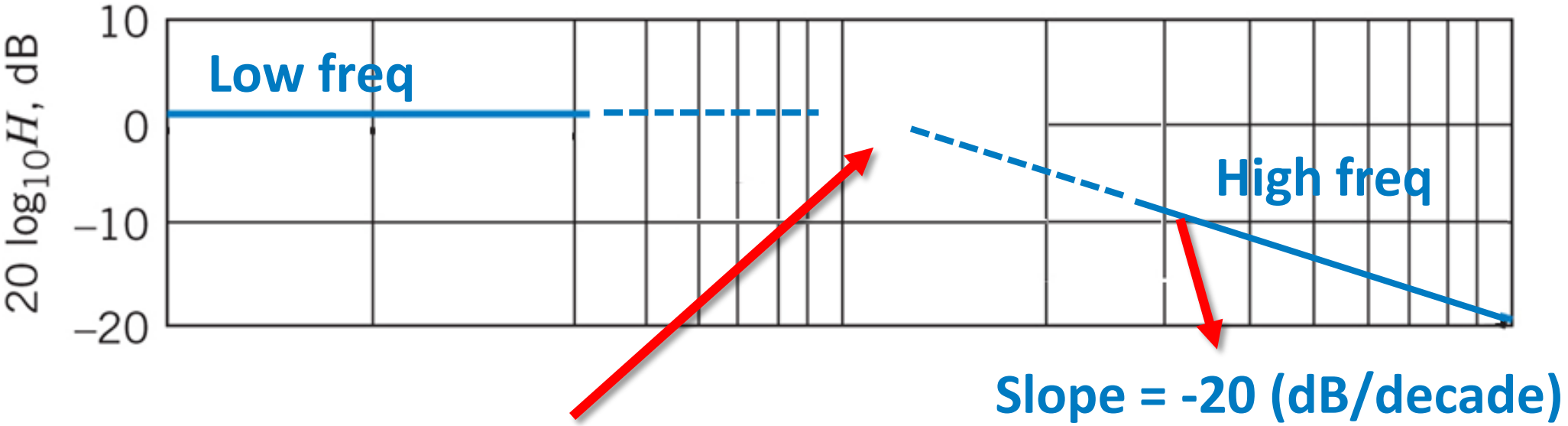
$$= -20\log_{10}\omega/\omega_0 = 20\log_{10}\omega_0 - 20\log_{10}\omega$$

- Reminding that the gain Bode plot shows $20\log_{10}H$ vs. $\log_{10}\omega$, the given high frequency asymptote is straight line!

Asymptotic Bode Plot

- We derived this:

$$20\log_{10}|\mathbf{H}(\omega)| = 20\log_{10}\frac{1}{\sqrt{1+(\omega/\omega_0)^2}} \cong \begin{cases} 0 & \text{Low freq} \\ 20\log_{10}\omega_0 - 20\log_{10}\omega & \text{High freq} \end{cases}$$



- **Important Question : Where do they intersect?**

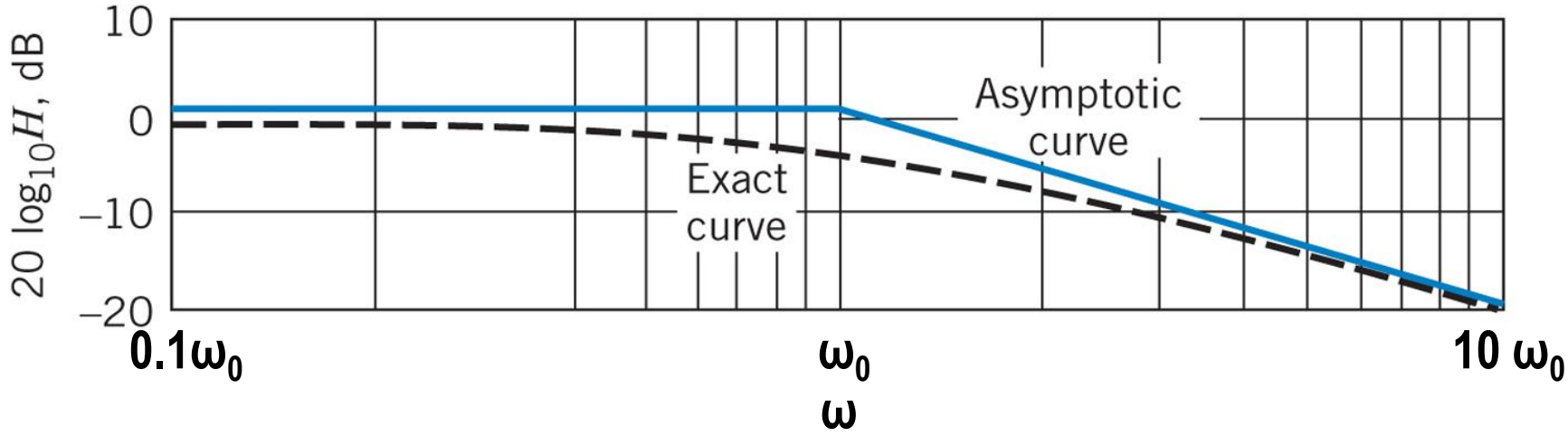
$$0 = 20\log_{10}\omega - 20\log_{10}\omega_0 \longrightarrow \omega = \omega_0$$

ω_0 is called "corner frequency"

Asymptotic Bode Plot; Near $\omega = \omega_0$

- Finally, we can obtain asymptotic bode plot as:

$$20\log_{10}|\mathbf{H}(\omega)| = 20\log_{10}\frac{1}{\sqrt{1+(\omega/\omega_0)^2}} \cong \begin{cases} 0 & \omega < \omega_0 \\ 20\log_{10}\omega_0 - 20\log_{10}\omega & \omega > \omega_0 \end{cases}$$



- Asymptotic Bode plot is a good approx. when $\omega << \omega_0$ or $\omega >> \omega_0$.
- Near $\omega = \omega_0$, asymptotic Bode plot deviates from the exact Bode plot.

➔ How much?

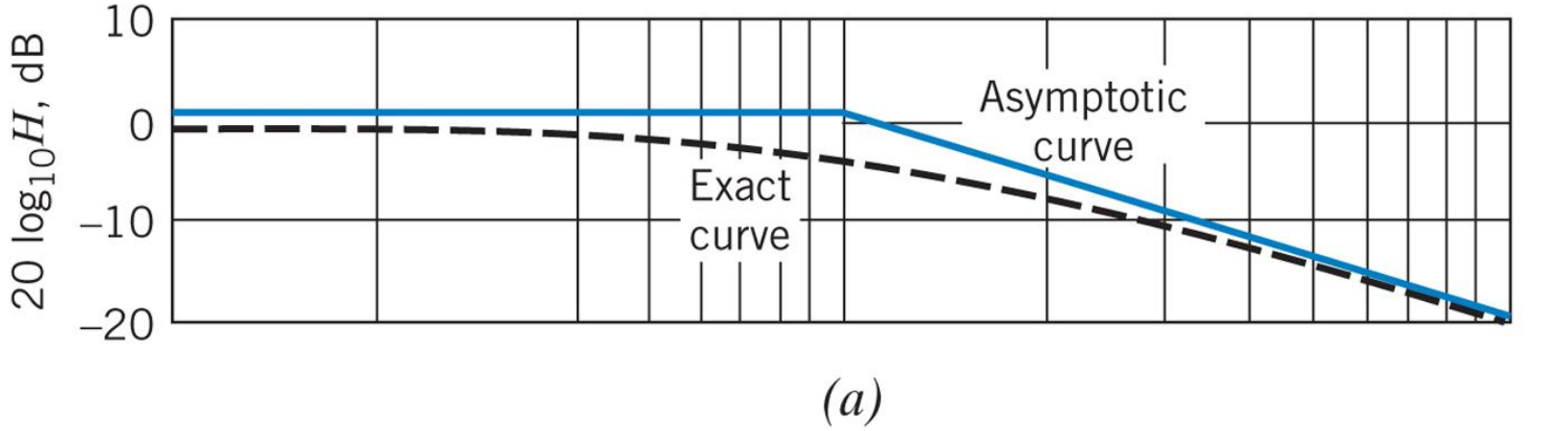
Bode Plot at $\omega = \omega_0$

- For the asymptotic Bode plot, the value becomes 0 when $\omega = \omega_0$.
- However, in the exact Bode plot,

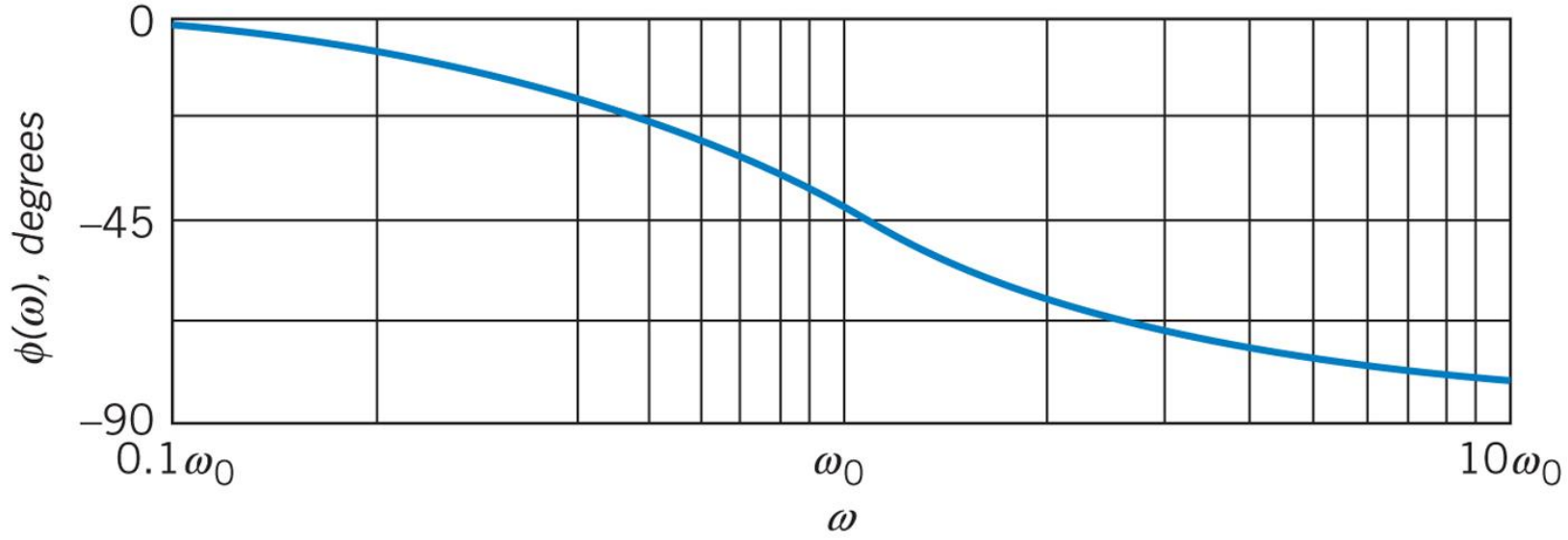
$$20\log_{10}|\mathbf{H}(\omega_0)| = 20\log_{10}\frac{1}{\sqrt{1+(\omega_0/\omega_0)^2}} = 20\log_{10}\frac{1}{\sqrt{2}} = -3.01\ \text{dB}$$

- At $\omega = \omega_0$, the magnitude drop is $1/\sqrt{2}$ in linear scale or -3 dB drop of logarithm scale.
- This frequency $\omega = \omega_0$ is called break frequency, corner frequency, etc.
  ➔ Often playing a role of key parameter of circuit.

Bode Plot for $H = {}^{1}/_{1+j\omega/\omega_0}$



(a)



(b)

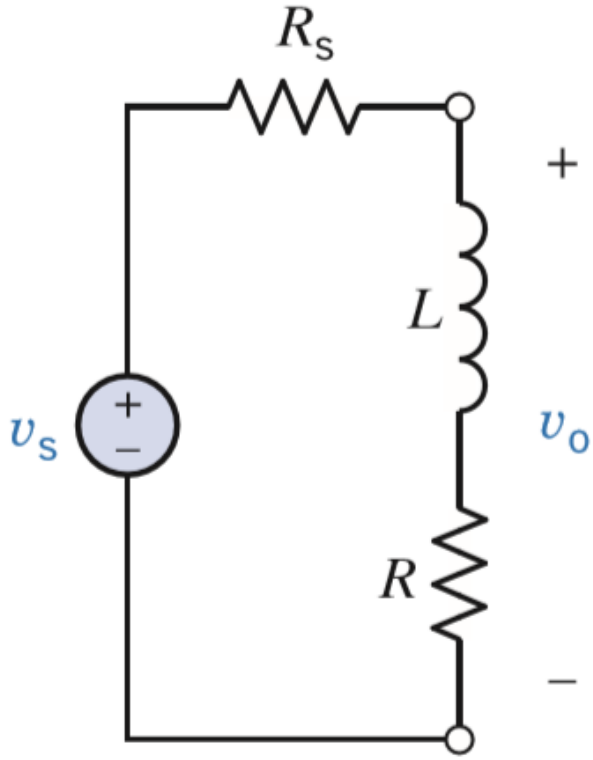
Bode Plot for Generalized H(ω)

- Network function of the shown circuit:

$$\mathbf{H} = \frac{\mathbf{V}_o}{\mathbf{V}_s} = \frac{R + j\omega L}{R_s + R + j\omega L}$$

- We can reform the above as:

$$\mathbf{H} = k\frac{1 + j\dfrac{\omega}{\omega_1}}{1 + j\dfrac{\omega}{\omega_2}} = H\angle\phi$$



- There are three key parameters : k, ω_1, ω_2

k = dc gain or low-frequency gain

ω_1 = zero

ω_2 = pole

Bode Plot for Generalized H(ω)

- Rewriting the given network function as,

$$\mathbf{H} = \frac{\mathbf{V}_o}{\mathbf{V}_s} = \frac{R + j\omega L}{R_s + R + j\omega L} \longrightarrow \mathbf{H} = k\frac{1 + j\dfrac{\omega}{\omega_1}}{1 + j\dfrac{\omega}{\omega_2}} = H\angle\phi$$

- Then,

$$\mathbf{H} = \left(\frac{R}{R + R_s}\right)\frac{1 + j\dfrac{\omega L}{R}}{1 + j\dfrac{\omega L}{R + R_s}}$$

$$k = \frac{R}{R + R_s}$$

$$\omega_1 = \frac{R}{L} < \frac{R + R_s}{L} = \omega_2$$

- Gain or magnitude of **H** is

$$H = k\frac{\sqrt{1 + \left(\dfrac{\omega}{\omega_1}\right)^2}}{\sqrt{1 + \left(\dfrac{\omega}{\omega_2}\right)^2}}$$

Obtaining Asymptotic Bode Plot

- Approximation for gain or magnitude

$$H = k\frac{\sqrt{1+\left(\frac{\omega}{\omega_1}\right)^2}}{\sqrt{1+\left(\frac{\omega}{\omega_2}\right)^2}}$$

$$\sqrt{1+(\omega/\omega_1)^2} \doteq \begin{cases} 1, \omega < \omega_1 \\ \frac{\omega}{\omega_1}, \omega > \omega_1 \end{cases}$$

$$\sqrt{1+(\omega/\omega_2)^2} \doteq \begin{cases} 1, \omega < \omega_2 \\ \frac{\omega}{\omega_2}, \omega > \omega_2 \end{cases}$$

$$H \cong \begin{cases} k & \omega < \omega_1 \\ \frac{k\omega}{\omega_1} & \omega_1 < \omega < \omega_2 \\ \frac{k\omega_2}{\omega_1} & \omega_2 < \omega \end{cases}$$

- Logarithmic gain

$$20\log_{10} H \cong \begin{cases} 20\log_{10} k & \omega < \omega_1 \\ (20\log_{10} k - 20\log_{10}\omega_1) + 20\log_{10}\omega & \omega_1 < \omega < \omega_2 \\ (20\log_{10} k - 20\log_{10}\omega_1) + 20\log_{10}\omega_2 & \omega_2 < \omega \end{cases}$$

Bode Plot with Normalized H

- Changing the value of k will shift the Bode plot up (increasing k) or down (decreasing k) but **will not change the shape** of the Bode plot.
- Normalizing the network function by dividing by the dc gain.

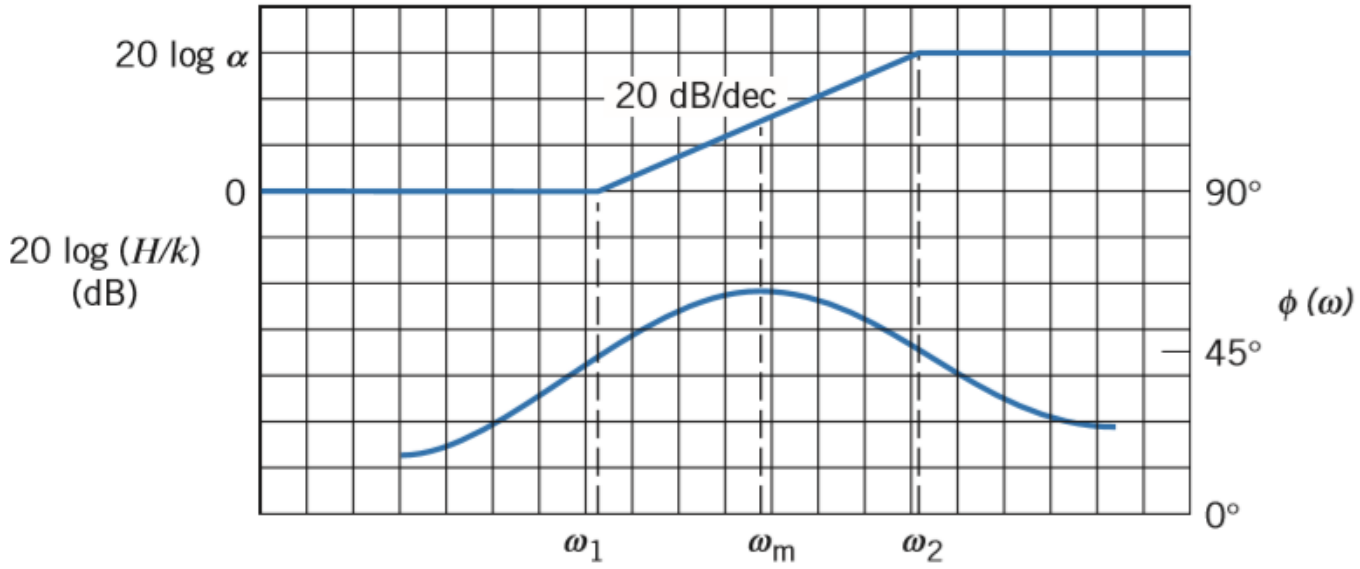
$$20\log_{10}\left(\frac{H}{k}\right) \cong \begin{cases} 0 & \omega < \omega_1 \\ 20\log_{10}\omega - 20\log_{10}\omega_1 & \omega_1 < \omega < \omega_2 \\ 20\log_{10}\omega_2 - 20\log_{10}\omega_1 & \omega_2 < \omega \end{cases}$$

- The phase angle of **H** is

$$\phi = \angle k + \underline{\angle\left(1 + j\frac{\omega}{\omega_1}\right)} - \underline{\angle\left(1 + j\frac{\omega}{\omega_2}\right)} = 0 + \tan^{-1}\left(\frac{\omega}{\omega_1}\right) - \tan^{-1}\left(\frac{\omega}{\omega_2}\right)$$

Bode Diagram

$$20 \log_{10}\left(\frac{H}{k}\right) \cong \begin{cases} 0 & \omega < \omega_1 \\ 20 \log_{10} \omega - 20 \log_{10} \omega_1 & \omega_1 < \omega < \omega_2 \\ 20 \log_{10} \omega_2 - 20 \log_{10} \omega_1 & \omega_2 < \omega \end{cases}$$



- Notice that:
  - Zeros cause the slope to increase by 20 dB/decade.
  - Poles cause the slope to decrease by 20 dB/decade.

Example 13.3-1

- Find the asymptotic magnitude Bode plot of

$$\mathbf{H}(\omega) = K \frac{j\omega}{1 + j\frac{\omega}{p}}$$