Chap. 14

The Laplace Transform (1)

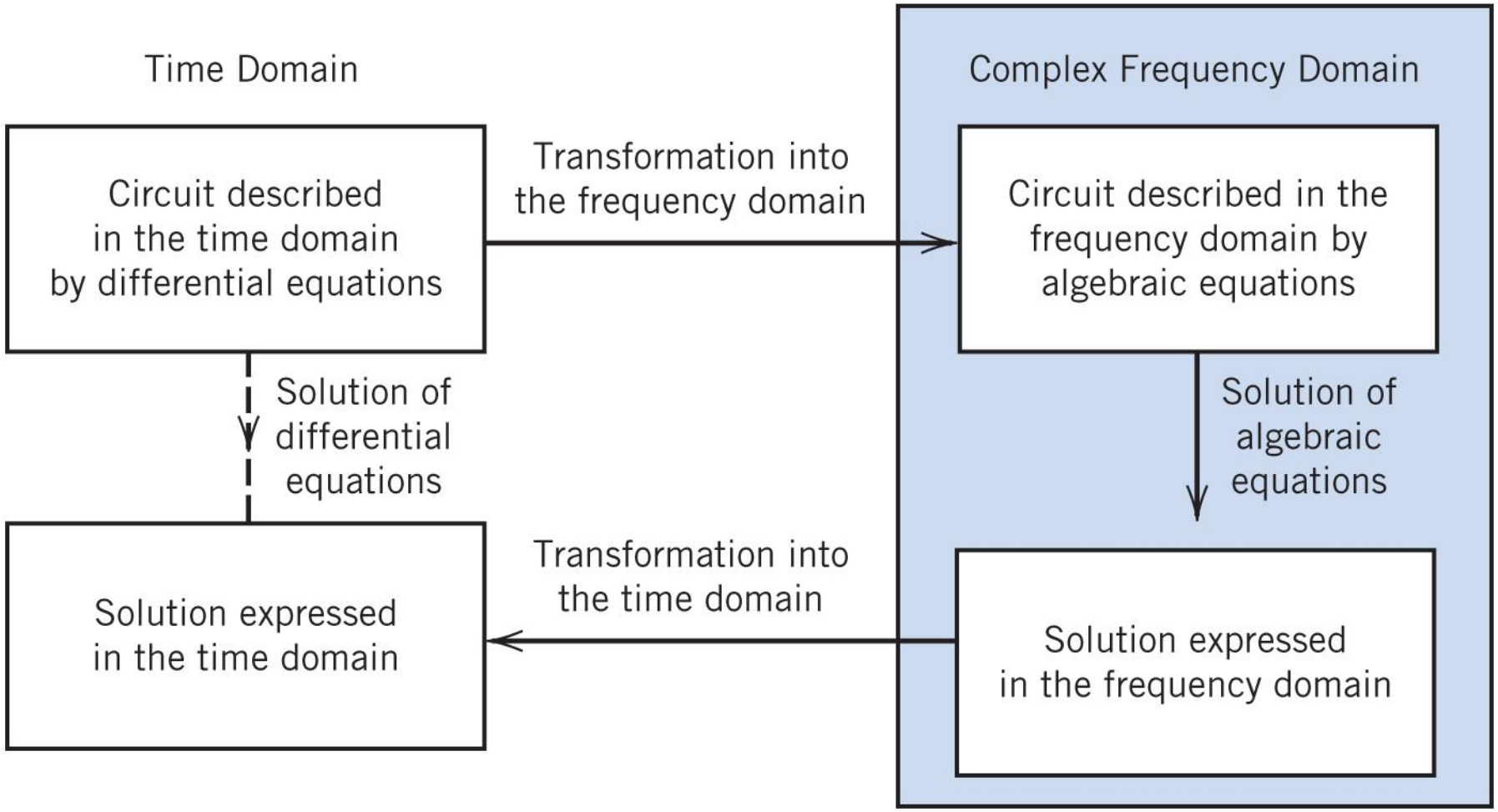
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Fundamental of Laplace Transform

- ✓ 14.2 Laplace Transform
- ✓ 14.3 Pulse Inputs
- ✓ 14.4 Inverse Laplace Transform
- ✓ 14.5 Initial and Final Value Theorem

Transform Method



Definition of Laplace Transform

- The Laplace transform is defined as

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

where s is a complex variable given by

$$s = \sigma + j\omega$$

- The exponent st of e must be dimensionless ➔ s = complex frequency
- The lower limit of the integral is 0-, a time just before t = 0.
- We say,

$$f(t) \leftrightarrow F(s)$$

Example 14.2-1

- Find the Laplace transform of f(t) = e^{-at} , where $a > 0$.
- Find the Laplace transform of g(t) = $e^{-at}u(t)$, where $a > 0$ and $u(t)$ is the unit step function.

Linearity of Laplace Transform

- Consider

$$f(t) = a_1 f_1(t) + a_2 f_2(t)$$

- We have

$$\begin{aligned} F(s) = \mathcal{L}[f(t)] = \mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] &= \int_{0-}^{\infty} (a_1 f_1(t) + a_2 f_2(t)) e^{-st} dt \\ &= a_1 \int_{0^-}^{\infty} f_1(t) e^{-st} dt + a_2 \int_{0^-}^{\infty} f_2(t) e^{-st} dt \\ &= a_1 F_1(s) + a_2 F_2(s) \end{aligned}$$

- To summarize,

$$a_1 f_1(t) + a_2 f_2(t) \quad \leftrightarrow \quad a_1 F_1(s) + a_2 F_2(s)$$

Example 14.2-2

- Find the Laplace transform of sin(ωt).

Differentiation in the Time Domain

- Consider

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_{0^-}^{\infty} \frac{df}{dt} e^{-st} dt$$

- Utilizing integrating by parts,

$$\mathcal{L}\left[\frac{df}{dt}\right] = s\int_{0}^{\infty} f e^{-st} dt + f e^{-st}\Big|_{0}^{\infty} = sF(s) - f(0^-)$$

- To summarize,

$$\frac{df}{dt} \leftrightarrow sF(s) - f(0^-)$$

Example 14.2-3

- Find the Laplace transform of cos(ωt).

Example 14.2-4

- Find the Laplace transform of $5 - 5e^{-2t}(1+2t)$

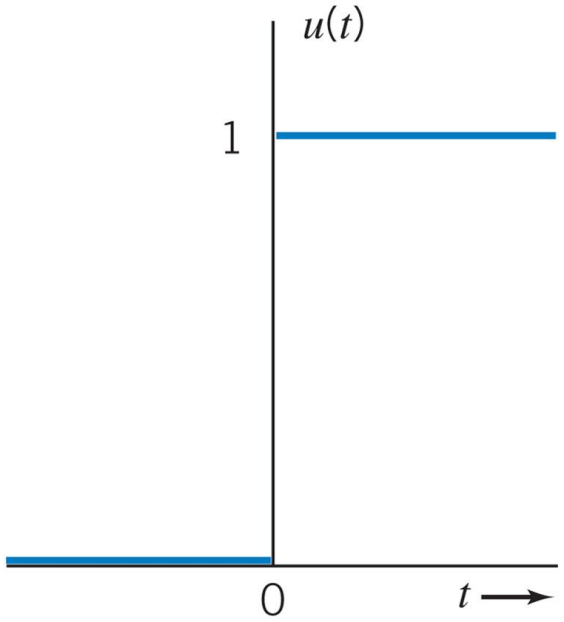
Example 14.2-5

- Find the Laplace transform of $10e^{-4t}\cos(20t + 36.9^{o})$

Pulse Inputs

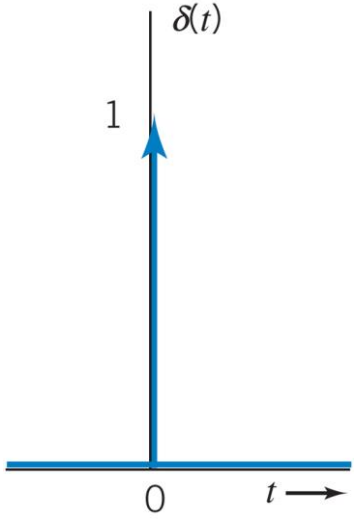
- Step function:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$



- Impulse function:

$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0 & t < 0 \\ undefined & t = 0 \\ 0 & t > 0 \end{cases}$$

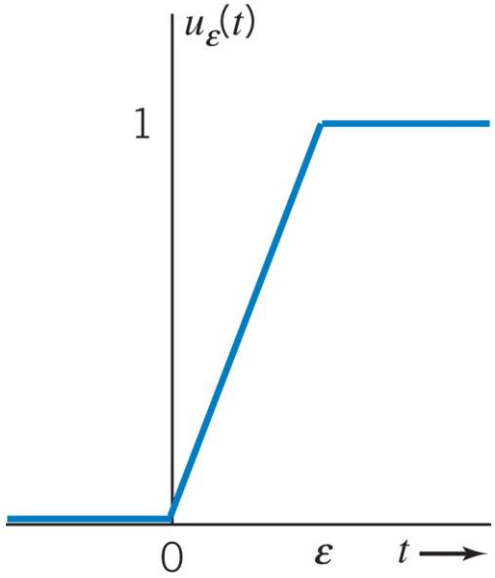


(d)

Approximate Pulse Inputs

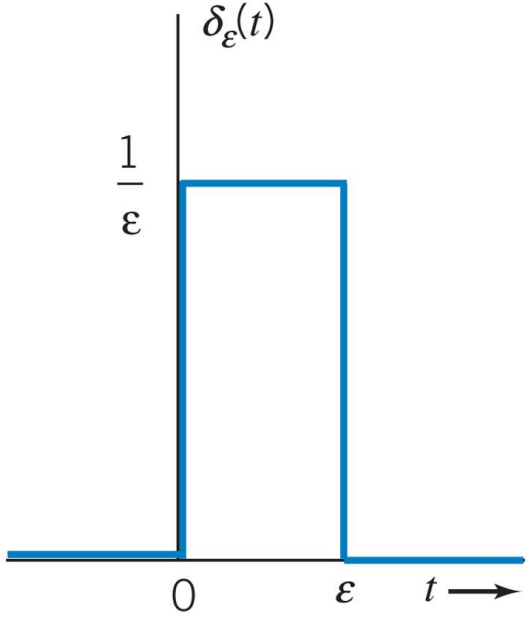
- We consider function which makes the transition from 0 to 1 over the time interval from 0 to ε.

$$\lim_{\varepsilon \to 0} u_\varepsilon(t) = u(t)$$



- Let

$$\delta_\varepsilon(t) = \frac{d}{dt} u_\varepsilon(t) = \begin{cases} 0 & t < 0 \\ \dfrac{1}{\varepsilon} & 0 < t < \varepsilon \\ 0 & t > \varepsilon \end{cases}$$



- Notice that for any value of ε,

$$\int_{-\infty}^{+\infty} \delta_\varepsilon(t)dt = \int_0^\varepsilon \frac{1}{\varepsilon} dt = 1$$

$$\delta(t) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t)$$

Sifting Property of δ(t)

- Sifting property of δ(t) at t = 0:

$$\int_{-\infty}^{+\infty} f(t)\delta(t)dt = f(0)$$

- Laplace transform of the impulse function:

$$\mathscr{L}[\delta(t)] = \int_{0-}^{\infty} e^{-st}\delta(t)dt = e^{0} = 1$$

Laplace TF of Time Shifted Function

- If time of f(t)u(t) is shifted by τ, $f(t-\tau)\ u(t-\tau)$
- Its Laplace transform is:

$$\mathscr{L}[f(t-\tau)u(t-\tau)] = \int_0^\infty f(t-\tau)u(t-\tau)e^{-st}dt = \int_\tau^\infty f(t-\tau)e^{-st}dt$$

- Let t - τ = x, then,

$$\mathscr{L}[f(t-\tau)u(t-\tau)] = \int_0^\infty f(x)e^{-s(\tau+x)}dx = e^{-s\tau}\int_\tau^\infty f(x)e^{-sx}dx = e^{-s\tau}F(s)$$

- To summarize,

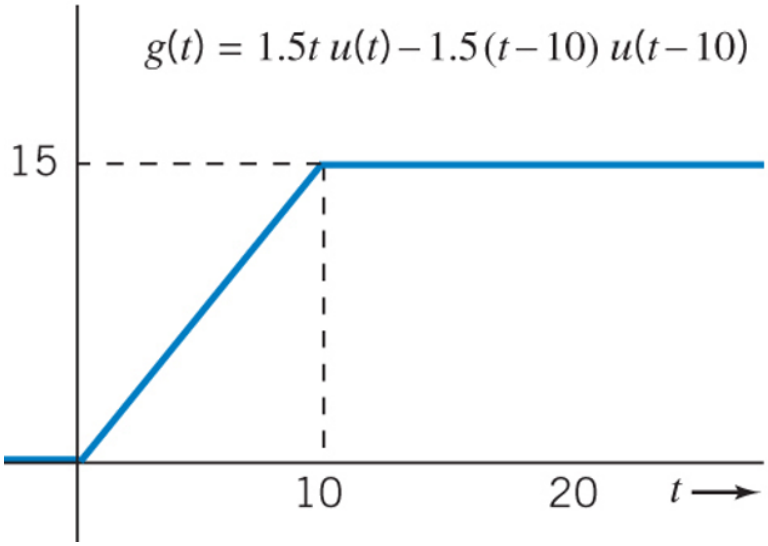
$$f(t-\tau)u(t-\tau) \leftrightarrow e^{-s\tau}F(s)$$

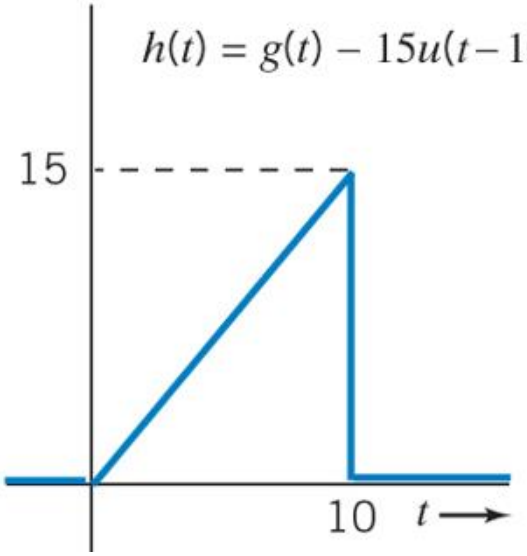
Example 14.2-6

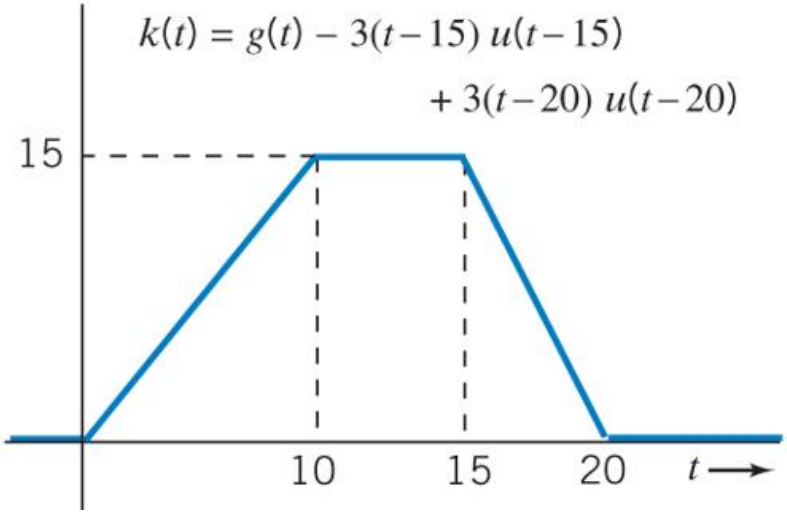
- Find the Laplace transform of $2\delta(t) + 3 + 4u(t)$.

Example 14.3-1

- Find the Laplace transform of g(t), h(t) and k(t)







Special Case of Leibniz Rule

- When a & b are constant,

$$\frac{d}{dx}\left(\int_a^b f(x,t)\,dt\right) = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt.$$

Differentiation in s domain

- Starting with the definition,

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

- Differentiating both side in s,

$$\frac{dF}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\, dt$$

$$= \int_0^\infty (-te^{-st}) f(t)\, dt = -\int_0^\infty e^{-st}[\, t\, f(t)\,]\, dt$$

$$= -\mathcal{L}\{t\, f(t)\}$$

- To summarize,

$$\mathcal{L}\{t\, f(t)\} = -\frac{dF}{ds} = -F'(s)$$

Laplace Transform Pairs

$f(t)$ for $t > 0$	$F(s) = \mathcal{L}[f(t)u(t)]$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{-at}t^n$	$\frac{n!}{(s+a)^{n+1}}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Laplace Transform Properties

PROPERTY	$f(t),\ t > 0$	$F(s) = \mathscr{L}[f(t)u(t)]$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Time scaling	$f(at)$, where $a > 0$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Time integration	$\int_0^t f(\tau) d\tau$	$\frac{1}{s} F(s)$
Time differentiation	$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2 f(t)}{dt^2}$	$s^2 F(s) - \left(sf(0^-) + \frac{df(0^-)}{dt}\right)$
	$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - \sum_{k=1}^{n} s^{n-k} \frac{d^{k-1} f(0^-)}{dt^{k-1}}$
Time shift	$f(t-a)u(t-a)$	$e^{-as} F(s)$
Frequency shift	$e^{-at} f(t)$	$F(s+a)$
Time convolution	$f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau) d\tau$	$F_1(s) F_2(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(\lambda) d\lambda$
Frequency differentiation	$tf(t)$	$-\frac{dF(s)}{ds}$
Initial value	$f(0^+)$	$\lim_{s \to \infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s \to 0} sF(s)$

Inverse Laplace Transform of F(s) with Simple Pole

- We want to derive Laplace TF of the following F(s):

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

- The roots of D(s) = 0 are called poles of F(s). Poles can be either real or complex (should exist in complex pair).

$$D(s) = s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = (s - p_1)(s - p_2) \cdots (s - p_n)$$

- Repeated pole? Simple pole?
- We can do a partial fraction expansion to express F(s):

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \frac{R_1}{s - p_1} + \frac{R_2}{s - p_2} + \cdots + \frac{R_i}{s - p_i} + \cdots + \frac{R_n}{s - p_n}$$

- The values of the residues of simple poles are calculated as:

$$R_i = (s - p_i) F(s)\big|_{s = p_i}$$

Example 14.4-1

- Find the inverse Laplace transform of $F(s) = \dfrac{s+3}{s^2+7s+10}$

F(s) with Simple Complex Conjugate Poles

- Suppose F(s) has $p_1 = -a+jb$ and $p_2 = -a-jb$.

$$F(s) = \frac{R_1}{s-p_1} + \frac{R_2}{s-p_2} + F_3(s) = \frac{c+jd}{s-(-a+jb)} + \frac{c-jd}{s-(-a-jb)} + F_3(s)$$

- We can combine the first two terms as:

$$F(s) = \frac{(c+jd)(s+a-jb)+(c-jd)(s+a-jb)}{(s+a-jb)(s+a+jb)} + F_3(s)$$

$$= \frac{2c\,s+2(a\,c-b\,d)}{s^2+2\,a\,s+a^2+b^2} + F_3(s) \quad = \frac{2\,c(s+a)-2\,b\,d}{(s+a)^2+b^2} + F_3(s)$$

$$= 2c\frac{s+a}{(s+a)^2+b^2} - 2d\frac{b}{(s+a)^2+b^2} + F_3(s)$$

- We can express F(s) as

$$F(s) = \frac{K_1 s + K_2}{s^2+2\,a\,s+a^2+b^2} + F_3(s)$$

F(s) with Simple Complex Conjugate Poles

- Taking the inverse Laplace transform for the first two terms,

$$\mathscr{L}^{-1}\left[2\,c\,\frac{s+a}{(s+a)^2+b^2}\right] = 2\,c\,\mathscr{L}^{-1}\left[\frac{s+a}{(s+a)^2+b^2}\right] = 2\,c\,e^{-at}\,\mathscr{L}^{-1}\left[\frac{s}{s^2+b^2}\right] = 2\,c\,e^{-at}\cos(bt)$$

and

$$\mathscr{L}^{-1}\left[2\,d\,\frac{b}{(s+a)^2+b^2}\right] = 2\,d\,\mathscr{L}^{-1}\left[\frac{b}{(s+a)^2+b^2}\right] = 2\,d\,e^{-at}\,\mathscr{L}^{-1}\left[\frac{b}{s^2+b^2}\right] = 2\,d\,e^{-at}\sin(bt)$$

- Finally, we have

$$\mathscr{L}^{-1}[F(s)] = 2\,c\,e^{-at}\cos(bt) - 2\,d\,e^{-at}\sin(bt) + \mathscr{L}^{-1}[F_3(s)]$$

Example 14.4-2

- Find the inverse Laplace transform of

$$F(s) = \frac{10}{(s^2 + 6s + 10)(s + 2)}$$

Example 14.4-2

F(s) with Repeated Poles

- F(s) has the repeated poles, which means

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{(s - p_1)^q (s - p_{q+1}) \cdots (s - p_n)}$$

- In this case, the partial fraction expansion of F(s) includes all powers of the term (s-p_1) up to q.

$$F(s) = \frac{R_1}{s - p_1} + \frac{R_2}{(s - p_1)^2} + \cdots + \frac{R_q}{(s - p_1)^q} + \frac{R_{q+1}}{s - p_{q+1}} + \cdots + \frac{R_n}{s - p_n}$$

- The residues corresponding to the repeated poles are

$$R_{q-k} = \frac{1}{k!} \left[\frac{d^k}{ds^k} (s - p_1)^q F(s) \right] \Bigg|_{s=p_1} \quad \text{for } k = q-1,\ q-2, \ldots, 2,\ 1,\ 0$$

- That is,

$$R_1 = \frac{1}{(q-1)!} \left[\frac{d^{q-1}}{ds^{q-1}} (s - p_1)^q F(s) \right] \Bigg|_{s=p_1},$$

$$R_2 = \frac{1}{(q-2)!} \left[\frac{d^{q-2}}{ds^{q-2}} (s - p_1)^q F(s) \right] \Bigg|_{s=p_1}, \ldots$$

$$R_q = [(s - p_1)^q F(s)]|_{s=p_1}$$

Example 14.4-3

- Find the Laplace Transform of

$$F(s) = \frac{4}{(s+1)^2(s+2)}$$

Example 14.4-3

Example 14.4-4
: Improper Rational Function

- Find the inverse Laplace transform of

$$F(s) = \frac{4\,s^3 + 15\,s^2 + s + 30}{s^2 + 5\,s + 6}$$

Initial Value Theorem

- Recall that

$$sF(s) - f(0-) = \mathcal{L}\left[\frac{df}{dt}\right] = \int_{0-}^{\infty} \frac{df}{dt} e^{-st} dt$$

- Taking the limit as s → ∞,

$$\lim_{s\to\infty} [sF(s) - f(0-)] = \lim_{s\to\infty} \int_{0-}^{0+} \frac{df}{dt} e^{-st} dt + \lim_{s\to\infty} \int_{0+}^{\infty} \frac{df}{dt} e^{-st} dt$$

- Thus,

$$\lim_{s\to\infty} sF(s) - f(0-) = f(0+) - f(0-)$$

- Then we can obtain the initial value theorem as:

$$f(0+) = \lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s)$$

Final Value Theorem

- Again,

$$sF(s) - f(0-) = \mathcal{L}\left[\frac{df}{dt}\right] = \int_{0-}^{\infty} \frac{df}{dt} e^{-st} dt$$

- How about taking the limit s → 0 to the above?

$$\lim_{s\to 0}[sF(s) - f(0-)] = \lim_{s\to 0}\int_{0}^{\infty}\left(\frac{df}{dt}\right)e^{-st}dt = \int_{0}^{\infty}\left(\frac{df}{dt}\right)e^{-0t}dt = f(\infty) - f(0-)$$

- Thus,

$$\lim_{s\to 0} sF(s) - f(0-) = f(\infty) - f(0-)$$

- Then we can obtain the final value theorem as:

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Example 14.5-1

- Consider the figure shown.
- Suppose that we know the following,

$$V(s) = \mathscr{L}[v(t)] = \frac{2s^2 + 30s + 136}{s(s^2 + 9s + 34)}$$

- Check whether the initial and final values of above v(t) agree with the figure

