Chap. 15

Fourier Series and Fourier Transform (1)

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Fourier Series

- ✓ 15.2 Fourier Series
- ✓ 15.3 Symmetry of the Function f(t)
- ✓ 15.4 Fourier Series of Selected Waveforms
- ✓ 15.5 Exponential Form of the Fourier Series
- ✓ 15.6 The Fourier Spectrum
- ✓ 15.7 Circuits and Fourier Series

Periodic Function, Period, and Fundamental Frequency

- The function f(t) is periodic if there exists a delay τ such that

$$f(t) = f(t - \tau)$$

- τ is **not unique**. Every integer multiple of τ also satisfies

$$f(t) = f(t - k\tau)$$

- The period T is defined as the smallest positive value of τ
- Then, the **fundamental frequency** ω_0 of f(t) is:

$$\omega_0 = \frac{2\pi}{T}$$

- Integer multiples of ω_0 are called harmonic frequencies.

Fourier Series

- A periodic function f(t) can be represented by an infinite series of harmonically related sinusoids:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\,\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\,\omega_0 t$$

- a_0 , a_n , b_n : Fourier trigonometric coefficients, which are

$$a_0 = \frac{1}{T}\int_{t_0}^{T+t_0} f(t)dt = \text{the average value of } f(t)$$

$$a_n = \frac{2}{T}\int_{t_0}^{T+t_0} f(t)\cos n\,\omega_0 t\,dt \quad n > 0$$

$$b_n = \frac{2}{T}\int_{t_0}^{T+t_0} f(t)\sin n\,\omega_0 t\,dt \quad n > 0$$

Revisiting $5\cos(\omega_0 t - 37^\circ)$

- Can you separate $5\cos(\omega_0 t - 37^\circ)$ into multiple trigonometric functions?

Dirichlet Conditions

1. $f(t)$ is a single-valued function except at possibly a finite number of points.
2. $f(t)$ is absolutely integrable, that is, $\int_{t_0}^{t_0+T} |f(t)|dt < \infty$ for any t_0.
3. $f(t)$ has a finite number of discontinuities within the period T.
4. $f(t)$ has a finite number of maxima and minima within the period T.

- We shall assume that the above are always satisfied for periodic voltage or current waveforms.

Steps for Obtaining Fourier Representation

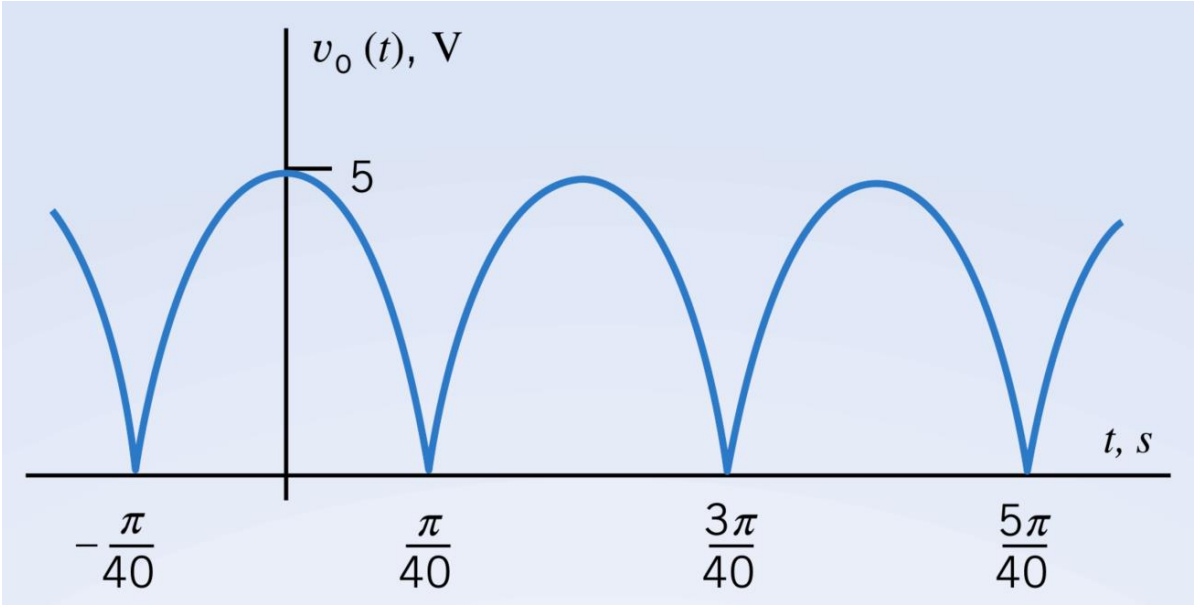
1) Determine the period T and the fundamental frequency ω_0 .
2) Represent the voltage or current waveform as a function of t over one complete period.
3) Determine the Fourier trigonometric coefficients a_0, a_n, b_n using integral expressions.
4) Substitute the coefficients a_0 , a_n, b_n obtained in Step 3) into

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\,\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\,\omega_0 t$$

Example 15.2-1

- Determine the Fourier series of $v_o(t)$.

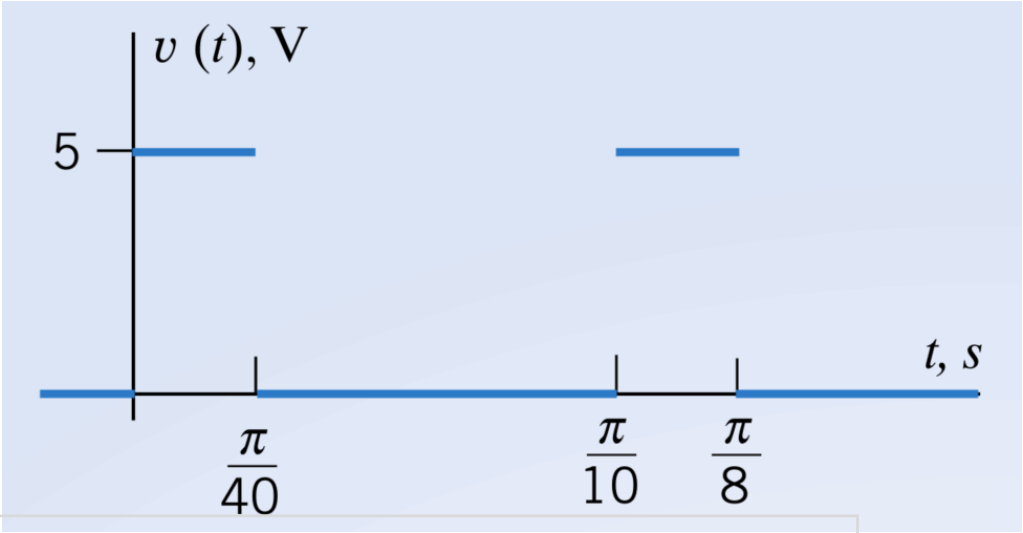




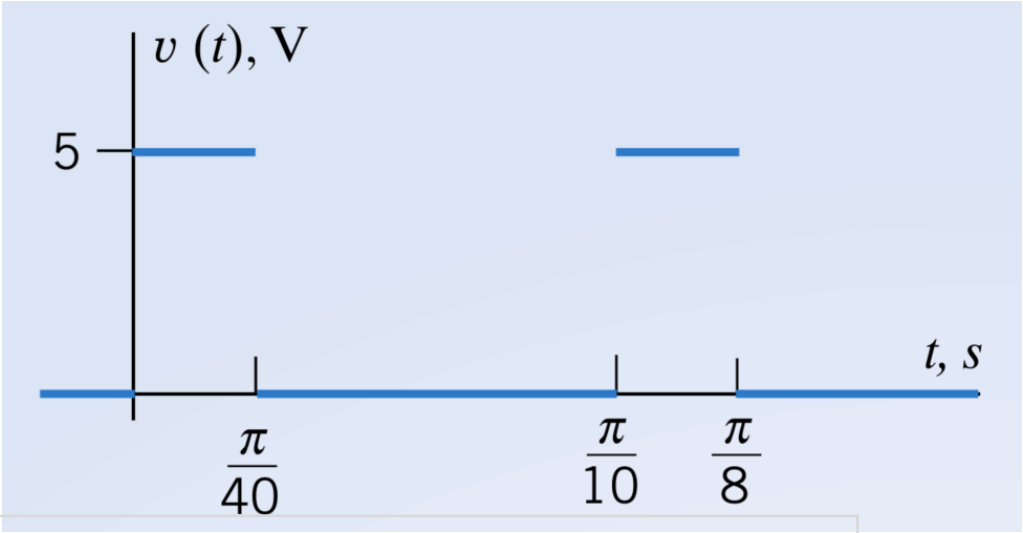

$v_i(t) = 5\cos 20t$ V
Full-Wave Rectifier
$v_o(t) = |v_i(t)|$

Example 15.2-2

- Determine the Fourier series of the pulse waveform shown.



Example 15.2-2



Alternate Representation of the Fourier series

- Fourier series:

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\,\omega_0 t + b_n \sin n\,\omega_0 t)$$

- Using $c_n = \sqrt{a_n^2 + b_n^2}$ and $\theta_n = -\tan^{-1}(b_n/a_n)$,

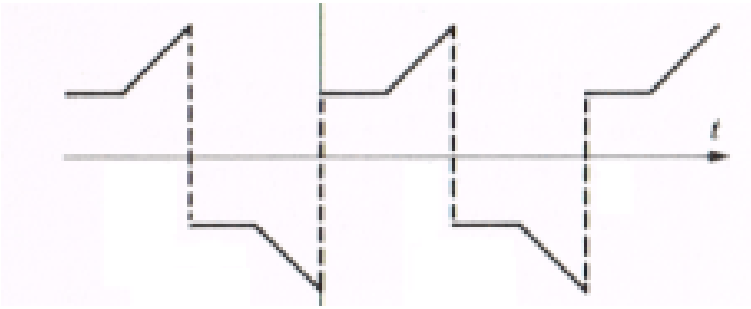
$$f(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\,\omega_0 t + \theta_n)$$

where $c_0 = a_0$.

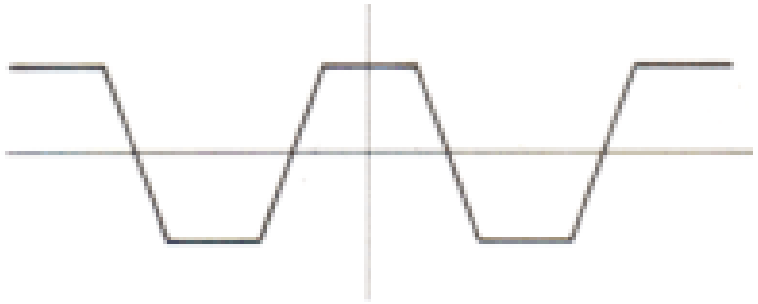
Symmetry of Function f(t)

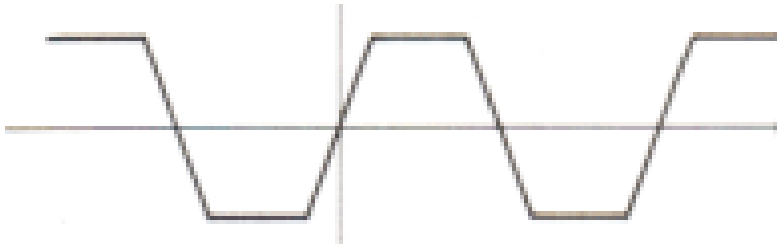
- Four types of symmetry:
  1) Even-function symmetry → $f(t) = f(-t)$
  2) Odd-function symmetry → $f(t) = -f(-t)$
  3) Half-wave symmetry

$$f(t) = -f\left(t - \frac{T}{2}\right)$$



  4) Quarter-wave symmetry
     = Half-wave symmetry + Even or Odd





Even/Odd Function

- Even : f(t) = f(-t), Odd : f(t) = -f(-t)
- Using the following,

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\,\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\,\omega_0 t$$

- Even : $b_n = 0$ for all n, and

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\,dt$$

- Odd : $a_n = 0$ for all n, and

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\,dt$$

Half-Wave Symmetry Function

- A function is half-wave symmetry if $f(t) = -f\left(t - \frac{T}{2}\right)$
- a_0 is zero. Why?
- For all n > 0,

$$a_n = \frac{2}{T}\int_{t_0}^{T+t_0} f(t)\cos n\,\omega_0 t\,dt \qquad b_n = \frac{2}{T}\int_{t_0}^{T+t_0} f(t)\sin n\,\omega_0 t\,dt$$

➔ $a_n = 0$ and $b_n = 0$ for all even values of n.

Quarter-wave Symmetry Function

- It is basically have-wave symmetry : $a_n = 0$, $b_n = 0$ for even n
- If function is odd,
  - $a_n = 0$ for all n
  - $b_n = 0$ for even n
  - b_n @ odd n can be non-zero and obtained as:

$$b_n = \frac{8}{T}\int_0^{T/4} f(t)\sin n\omega_0 t\, dt$$

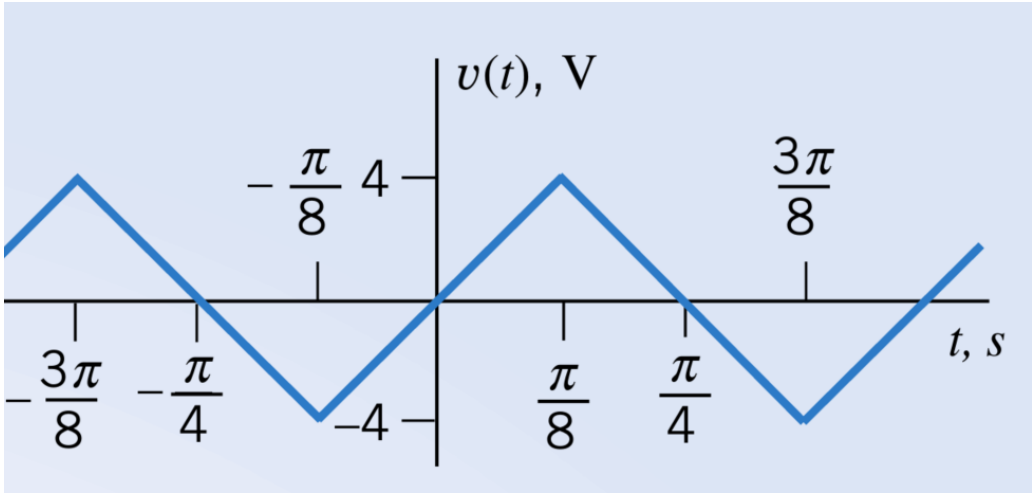
- If function is even,
  - $b_n = 0$ for all n
  - $a_n = 0$ for even n
  - a_n @ odd n can be non-zero and obtained as:

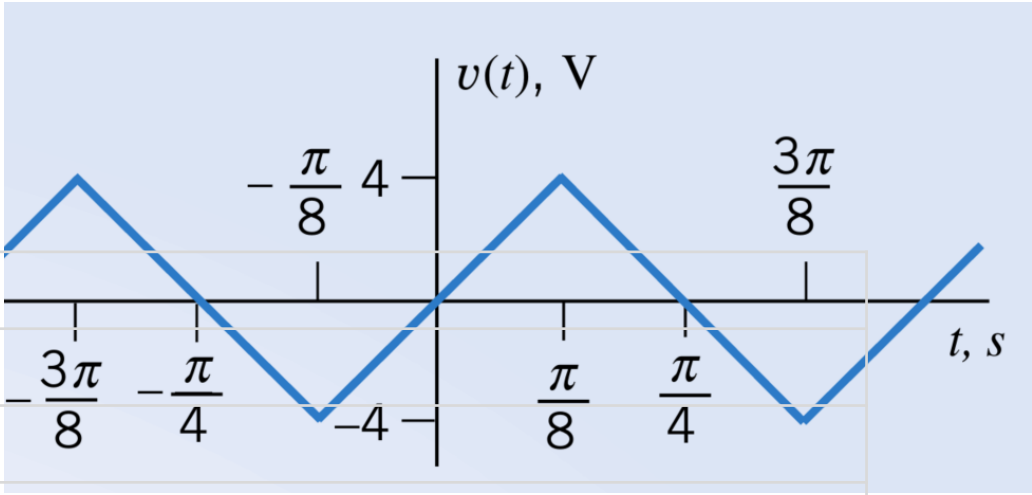
$$a_n = \frac{8}{T}\int_0^{T/4} f(t)\cos n\omega_0 t\, dt$$

SYMMETRY	FOURIER COEFFICIENTS
1. Odd function $f(t) = -f(-t)$	$a_n = 0$ for all n $b_n = \frac{4}{T}\int_0^{T/2} f(t) \sin n\omega_0 t \, dt$
2. Even function $f(t) = f(-t)$	$b_n = 0$ for all n $a_n = \frac{4}{T}\int_0^{T/2} f(t) \cos n\omega_0 t \, dt$
3. Half-wave symmetry $f(t) = -f\left(t - \frac{T}{2}\right)$	$a_0 = 0$ $a_n = 0$ for even n $b_n = 0$ for even n $a_n = \frac{4}{T}\int_0^{T/2} f(t) \cos n\omega_0 t \, dt$ for odd n $b_n = \frac{4}{T}\int_0^{T/2} f(t) \sin n\omega_0 t \, dt$ for odd n
4. Quarter-wave symmetry. (Half-wave symmetry and symmetry about the midpoints of the positive and negative half cycles)	A. Odd function: $a_0 = 0$, $a_n = 0$ for all n $b_n = 0$ for even n $b_n = \frac{8}{T}\int_0^{T/4} f(t) \sin n\omega_0 t \, dt$ for odd n B. Even function: $a_0 = 0$, $b_n = 0$ for all n $a_n = 0$ for even n $a_n = \frac{8}{T}\int_0^{T/4} f(t) \cos n\omega_0 t \, dt$ for odd n

Example 15.3-1

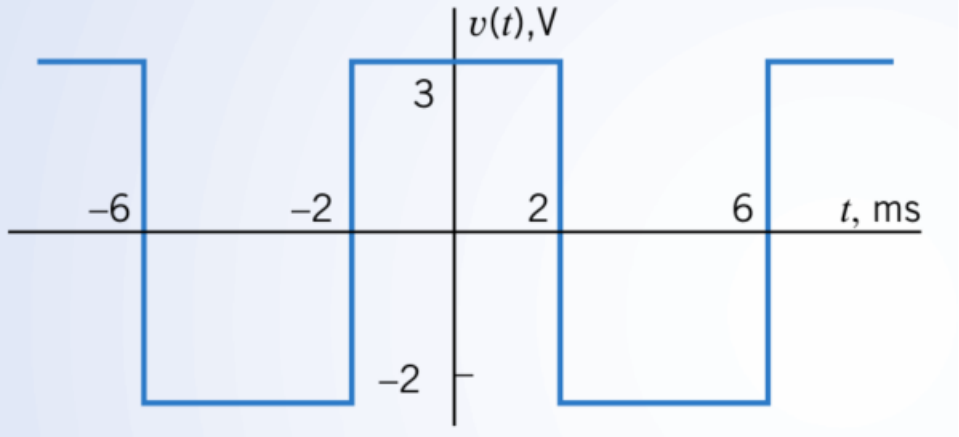
- Determine the Fourier series of the waveform v(t) shown.




v(t), V
4
−4
−3π/8
−π/4
−π/8
π/8
π/4
3π/8
t, s

Example 15.4-1

- Determine the Fourier series of the waveform shown.



Exponential Form of the Fourier Series

- Recall that $f(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \theta_n)$
- Using $\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$,

$$f(t) = c_0 + \sum_{n=1}^{\infty} \frac{c_n}{2}\left(e^{j(n\omega_0 t+\theta_n)} + e^{-j(n\omega_0 t+\theta_n)}\right)$$

$$= c_0 + \sum_{n=1}^{\infty}\left(\frac{c_n}{2}e^{j\theta_n}\right)e^{jn\omega_0 t} + \sum_{n=1}^{\infty}\left(\frac{c_n}{2}e^{-j\theta_n}\right)e^{-jn\omega_0 t}$$

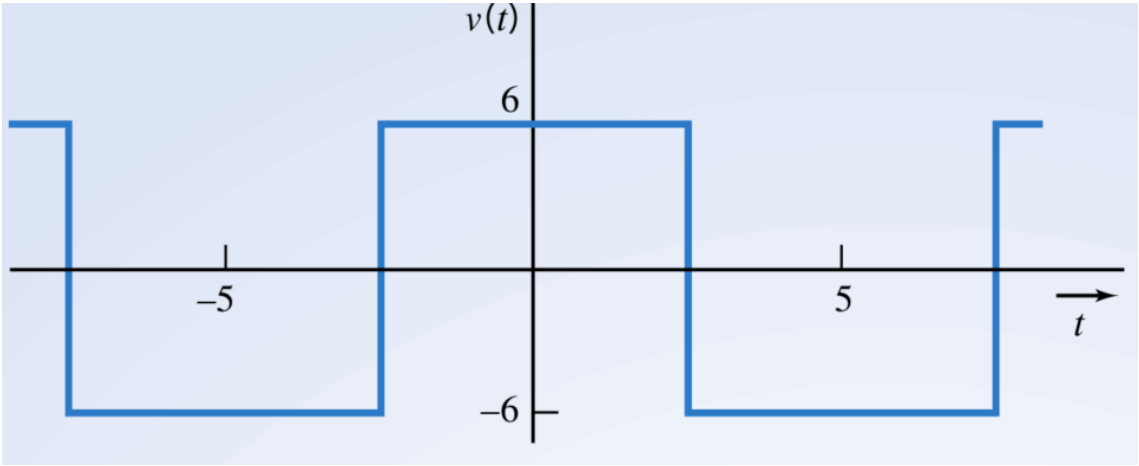
- With defining

$$C_0 = c_0 e^{j0} = \mathbf{C}_0 \qquad \mathbf{C}_n = \frac{c_n}{2}e^{j\theta_n}, \quad \text{and} \quad \mathbf{C}_{-n} = \frac{c_n}{2}e^{-j\theta_n}$$

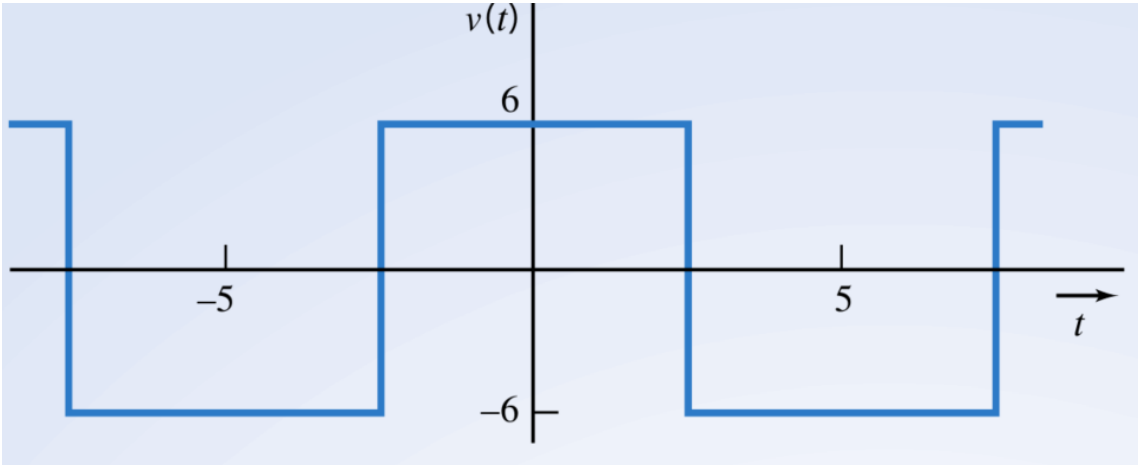
- Fourier series can be written as

$$f(t) = \sum_{n=-\infty}^{\infty} \mathbf{C}_n e^{jn\omega_0 t} \qquad \mathbf{C}_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-jn\omega_0 t}dt$$

Example 15.5-1



- Determine the exponential Fourier series for the function shown.


v(t)
6
–6
–5
5
t

Fourier Spectrum

- The **Fourier spectrum** is a graphical display of the amplitude and phase of the complex Fourier coefficients at the fundamental and harmonic frequencies.

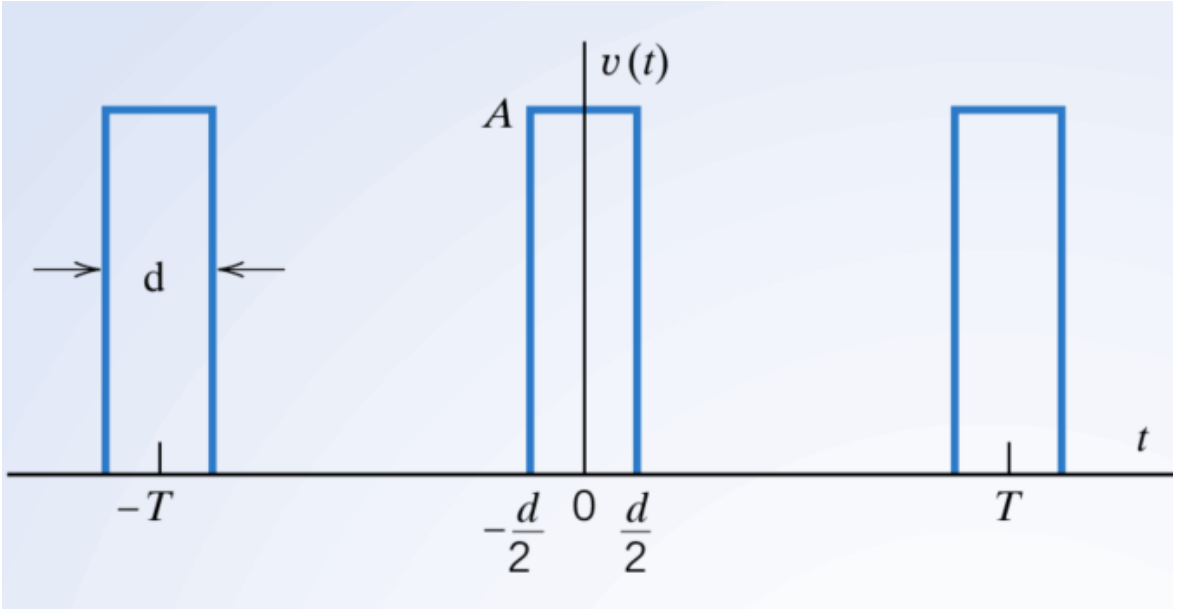
- We can plot the complex C_n as a function of angular frequency $\omega = n\omega_0$.

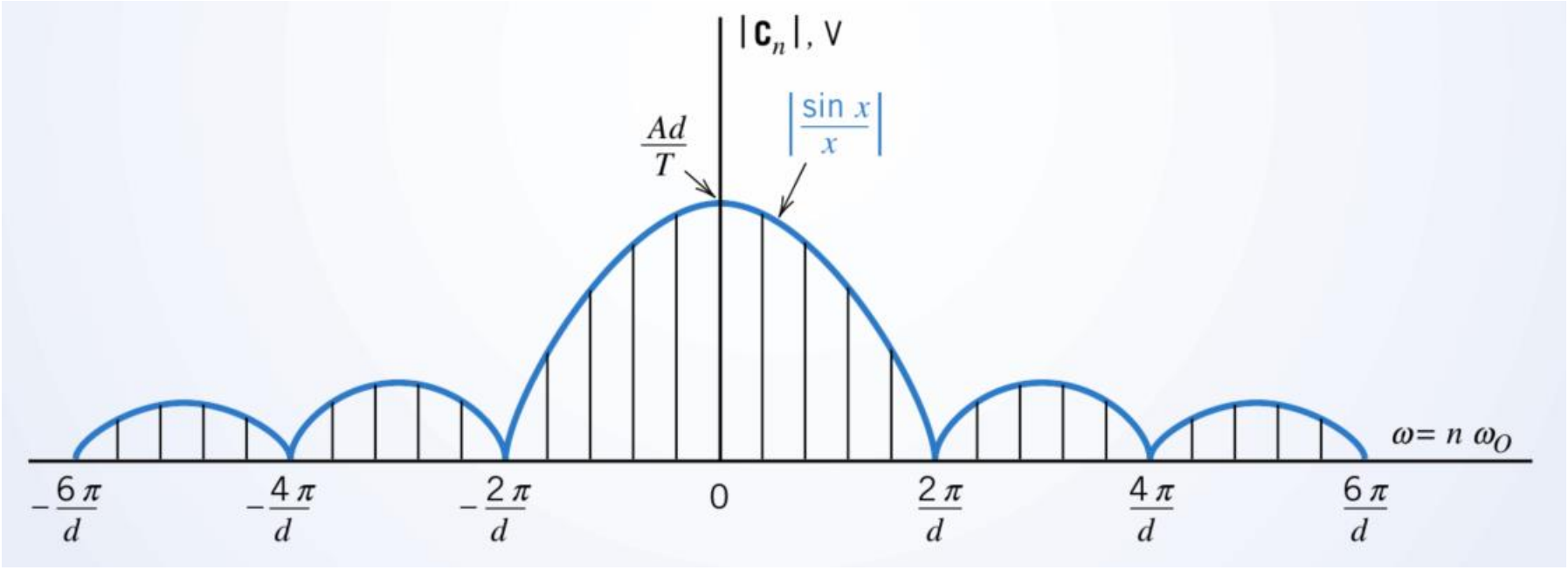
$$\mathbf{C}_n = |\mathbf{C}_n| \angle \theta_n$$

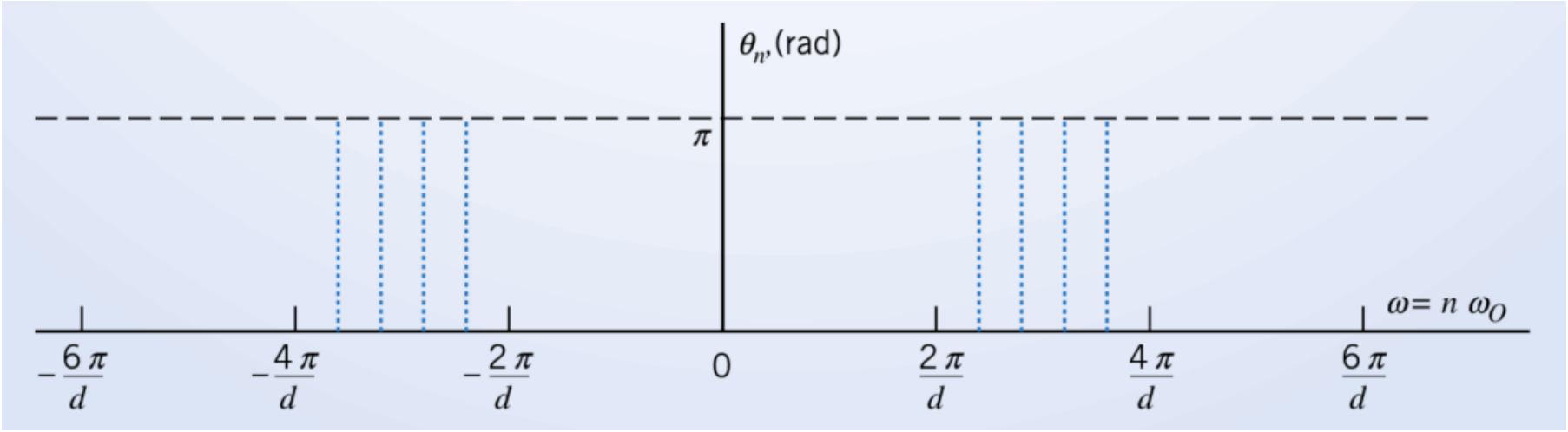
- How can we plot complex? Plotting $|C_n|$ and $\angle\theta_n$ separately ➔ **Amplitude spectrum & Phase spectrum,** respectively

Example 15.6-1

- Determine the Fourier spectrum



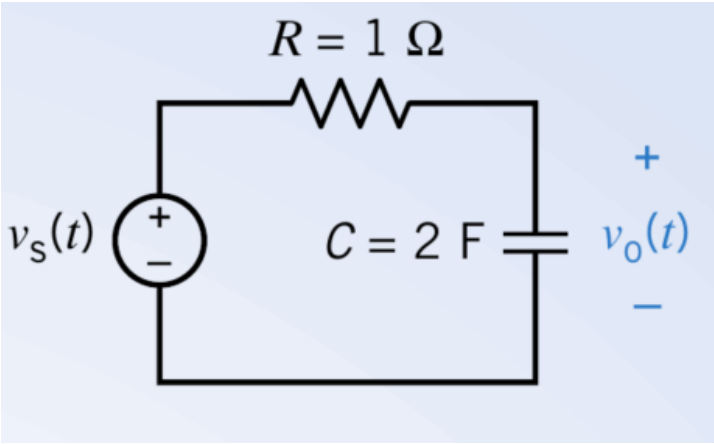

$|C_n|$, V
$\frac{Ad}{T}$
$\left|\frac{\sin x}{x}\right|$
$\omega = n\,\omega_O$
$-\frac{6\pi}{d}$
$-\frac{4\pi}{d}$
$-\frac{2\pi}{d}$
0
$\frac{2\pi}{d}$
$\frac{4\pi}{d}$
$\frac{6\pi}{d}$


θ_n,(rad)
π
$\omega = n\,\omega_O$
$-\frac{6\pi}{d}$
$-\frac{4\pi}{d}$
$-\frac{2\pi}{d}$
0
$\frac{2\pi}{d}$
$\frac{4\pi}{d}$
$\frac{6\pi}{d}$

Circuits and Fourier Series

- When we determine the circuit response excited by a periodic input signal $v_s(t)$, we can utilize Fourier series.
- In linear circuit, we can consider that the total response is the sum of the response.

Example 15.7-1



- Find the steady-state response $v_o(t)$ of the RC circuit shown. The input $v_s(t)$ is the square wave shown.
- Use first four terms of Fourier series

