# Chap. 15

## Fourier Series and Fourier Transform (2)

Hanwool Jeong

hwjeong@kw.ac.kr

# Fourier Transform

- ✓ 15.9 Fourier Transform
- ✓ 15.10 Fourier Transform Properties
- ✓ 15.11 The Spectrum of Signals
- ✓ 15.12 Convolution and Circuit Response
- ✓ 15.13 The Fourier Transform and the Laplace Transform

# What Really Means in Fourier Series

- Recall the Fourier Series:

$$f(t) = \sum_{n=-\infty}^{n=\infty} C_n e^{jn\omega_0 t} \qquad C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t}\, dt$$

- $C_n$ is meaningful because it can tell how much f(t) contains information at each frequency components.

# Infinity Period in Fourier Series

- Recall the Fourier series:

$$f(t) = \sum_{n=-\infty}^{n=\infty} C_n e^{jn\omega_0 t} \qquad C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t}\, dt$$

- By increasing the period T to infinity, we can extend the above into non-periodic signal.
- As T $\rightarrow\infty$,
  - The difference b/w two consecutive harmonic frequencies, $\Delta\omega = (n+1)\omega_0 - n\omega_0 = \omega_0 =$ **$2\pi/T$ becomes $d\omega$**.
  - $n\omega_0$ approaches the continuous variable $\omega$

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t}\, dt \quad \Rightarrow \quad C(\omega) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\omega t} dt$$

# Fourier Transform

- Usually, we are interested with time confined signal,

$$\lim_{T\to\infty}\int_{-T/2}^{T/2} f(t)e^{-j\omega t}dt < \infty$$

- Thus, let's consider $C(\omega)T$ instead of $C(\omega)$ as the indicator for the information quantity for various frequencies.
- With infinity T, as a new frequency function $F(\omega)$:

$$C(\omega)\times T = \lim_{T\to\infty}\int_{-T/2}^{T/2} f(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = F(\omega)$$

- ***F*(*ω*) is defined as the Fourier transform of *f*(t).**

# Inverse Fourier Transform

- Inverse Fourier Transform is the process deriving f(t) from *F*(ω)
- Recall the Fourier Series:

$$f(t) = \sum_{n=-\infty}^{n=\infty} C_n e^{jn\omega_0 t} \Rightarrow f(t) = \lim_{T\to\infty} \sum C(\omega) e^{j\omega t} = \lim_{T\to\infty} \frac{1}{T} \sum T \times C(\omega) e^{j\omega t}$$

$$= \lim_{T\to\infty} \frac{1}{T} \sum F(\omega) e^{j\omega t}$$

- With T = 2π/dω

$$f(t) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

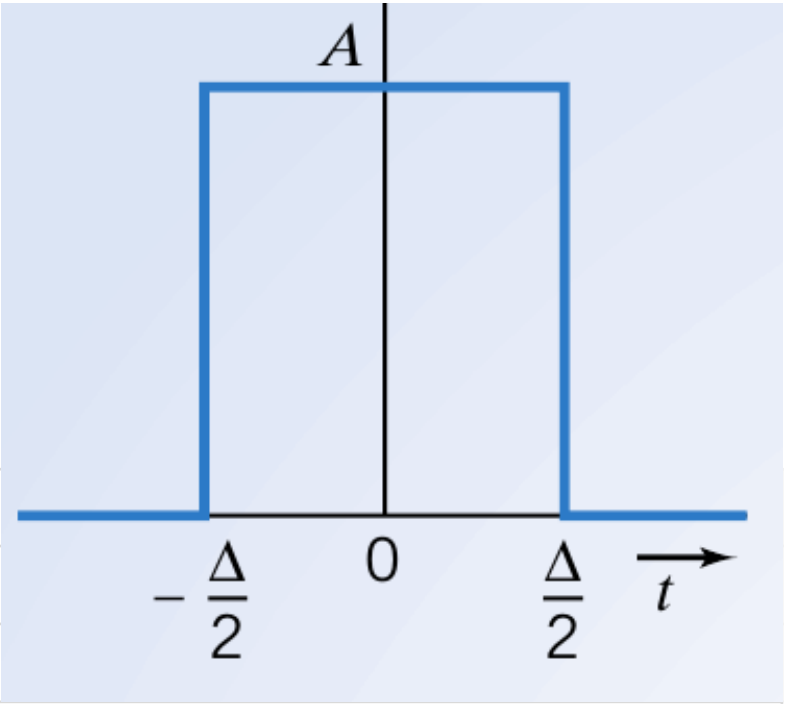
# Summary

**Table 15.9-1 The Fourier Transform Pair**

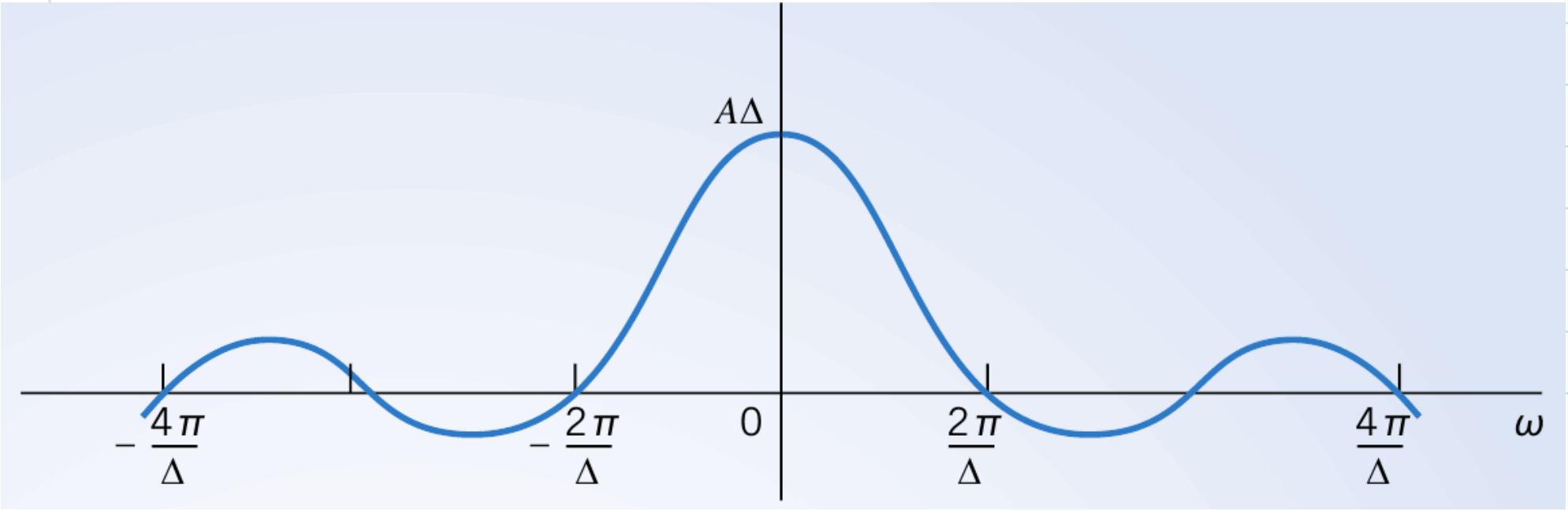
| EQUATION | NAME | PROCESS |
|---|---|---|
| $F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$ | Transform | Time domain to frequency domain<br>Conversion of $f(t)$ into $F(j\omega)$ |
| $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t}\, d\omega$ | Inverse transform | Frequency domain to time domain<br>Conversion of $F(j\omega)$ into $f(t)$ |

# Example 15.9-1

- Determine Fourier Transform of shown pulse.



# Example 15.9-1



# Linearity Property

$$\begin{aligned}\mathscr{F}[af_1 + bf_2] &= \int_{-\infty}^{\infty} [af_1 + bf_2]e^{-j\omega t}\,dt \\ &= \int_{-\infty}^{\infty} af_1 e^{-j\omega t}dt + \int_{-\infty}^{\infty} bf_2 e^{-j\omega t}dt \\ &= aF_1(\omega) + bF_2(\omega)\end{aligned}$$

# Example 15.10-1

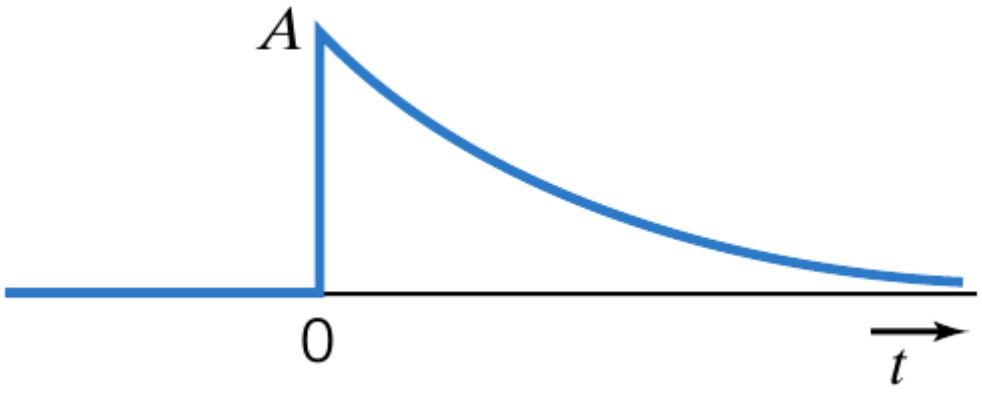
- Find the Fourier transform of a time-shifted function $f(t-t_0)$.

# Example

- Derive the Fourier transform of the follwoing in terms of $F(\omega)$, when $\mathcal{F}[f(t)] = F(\omega)$.

$$e^{j\omega_0 t} f(t)$$

# Example




- Find the Fourier transform of $f(t) = Ae^{-at}u(t)$

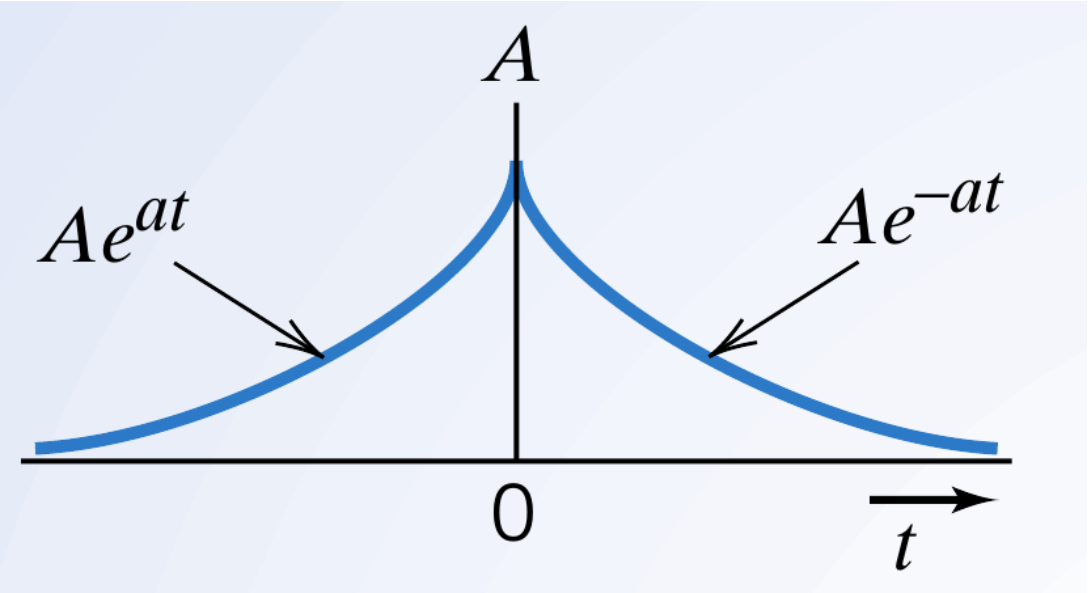
# Example

- Find $\mathcal{F}$[f(-t)] when $\mathcal{F}$[f(t)] = $F(\omega)$.

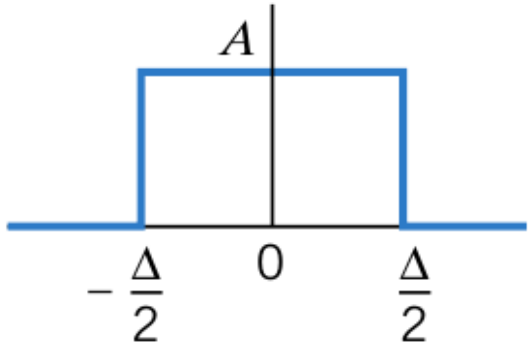
| FUNCTION OF TIME | FOURIER TRANSFORM |
|---|---|
| $f(t)$ | $F(\omega)$ |
| $Af(t)$ | $AF(\omega)$ |
| $af_1 + bf_2$ | $aF_1(\omega) + bF_2(\omega)$ |
| $f(t - t_0)$ | $e^{-j\omega t_0}F(\omega)$ |
| $f(at), a > 0$ | $\frac{1}{a}F\left(\frac{\omega}{a}\right)$ |
| $e^{j\omega_0 t}f(t)$ | $F(\omega - \omega_0)$ |
| $\frac{d^n f(t)}{dt^n}$ | $(j\omega)^n F(\omega)$ |
| $\int_{-\infty}^{\infty} f_1(x)f_2(t - x)dx$ | $F_1(\omega)F_2(\omega)$ |
| $t^n f(t)$ | $(j)^n \frac{d^n F(\omega)}{d\omega^n}$ |
| $f(-t)$ | $F(-\omega)$ |
| $\int_{-\infty}^{t} f(\tau)\, d\tau$ | $\frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$ |

# Example 15.10-2

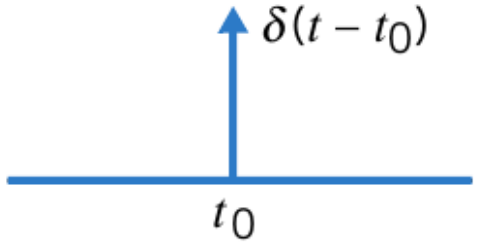
- Find the Fourier transform of $f(t) = Ae^{-a|t|}$



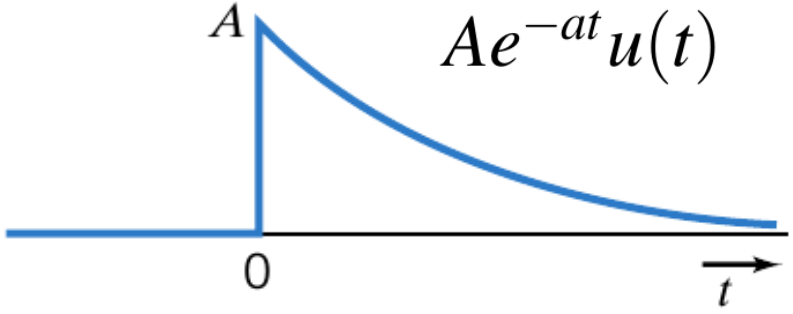
$$f_1(t) = Au\left(t + \frac{\Delta}{2}\right) - Au\left(t - \frac{\Delta}{2}\right)$$



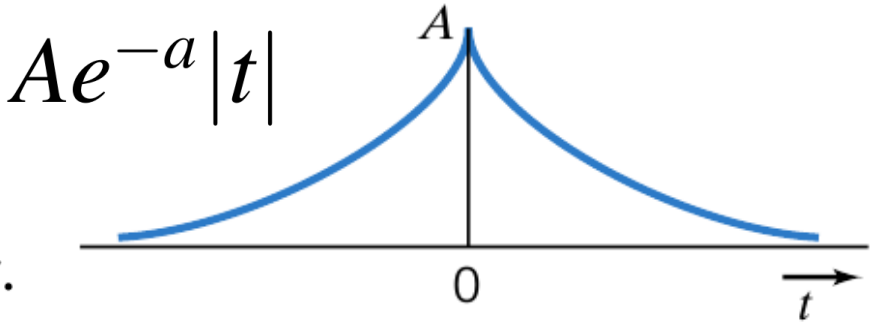

$$A\Delta Sa\left(\frac{\omega\Delta}{2}\right)$$




$$e^{-j\omega t_0}$$




$$\frac{A}{a + j\omega}$$

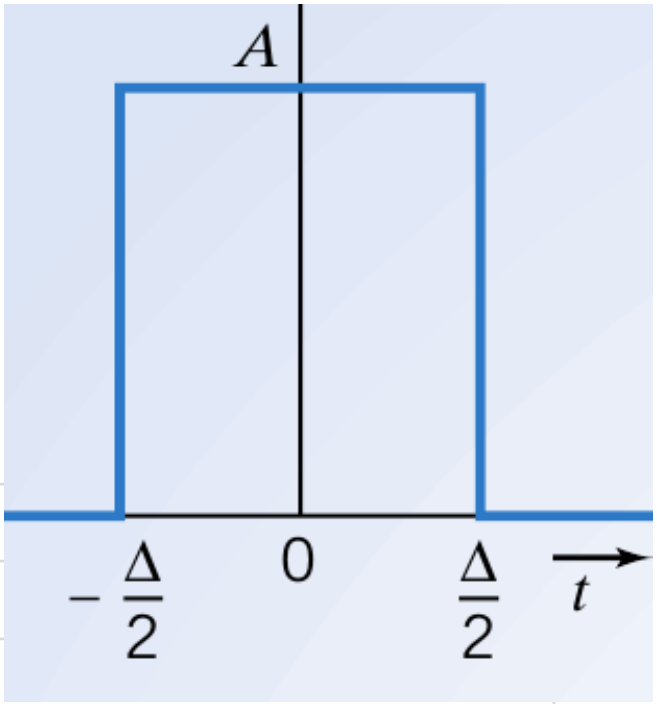



$$\frac{2aA}{a^2 + \omega^2}$$

*Note:* $Sa(x) = (\sin x)/x$.

# Example 15.10-3

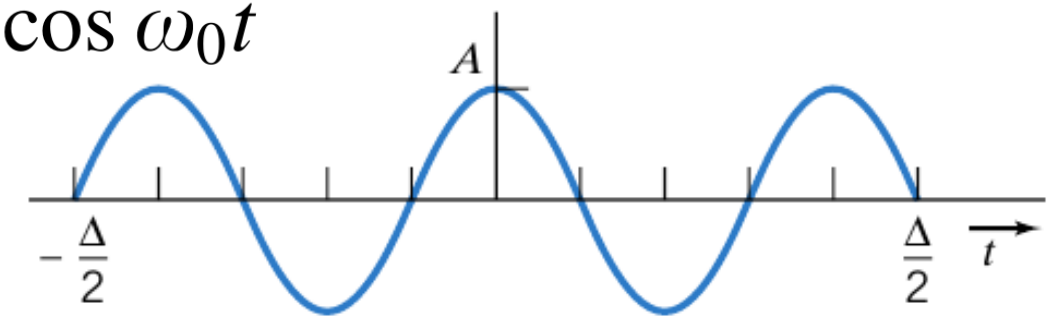
$f_1(t)$



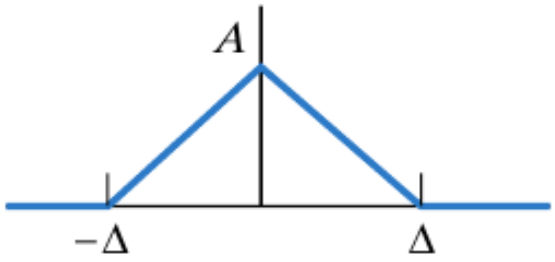

- Find the Fourier transform of the gated cosine waveform $f(t) = f_1(t)\cos\omega_0 t$, where $f_1(t)$ is the rectangular pulse shown.

# Example 15.10-3

$Af_1(t)\cos\omega_0 t$

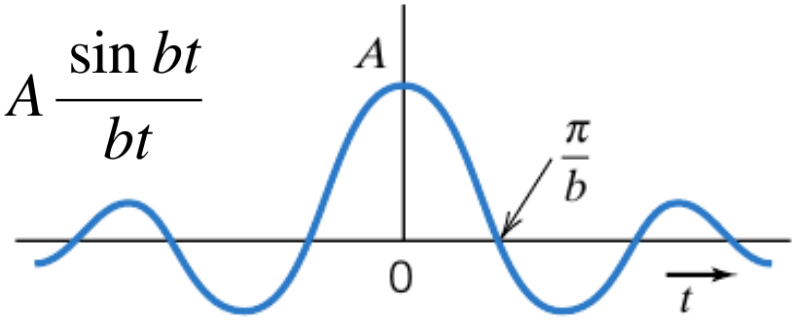



$$\frac{A\Delta}{2}[Sa(\omega-\omega_0)+Sa(\omega+\omega_0)]$$

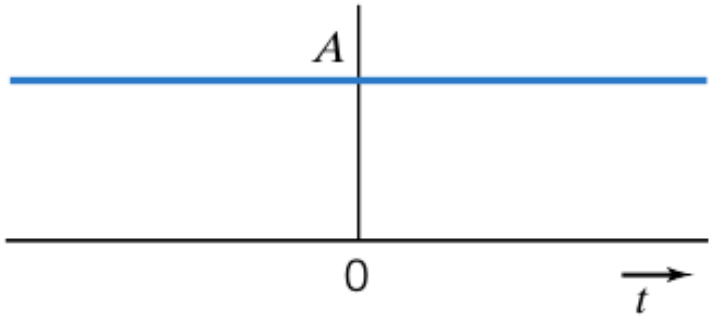



$$A\,\Delta Sa^2\left(\frac{\omega\Delta}{2}\right)$$

$$A\,Sa(bt)=A\,\frac{\sin bt}{bt}$$

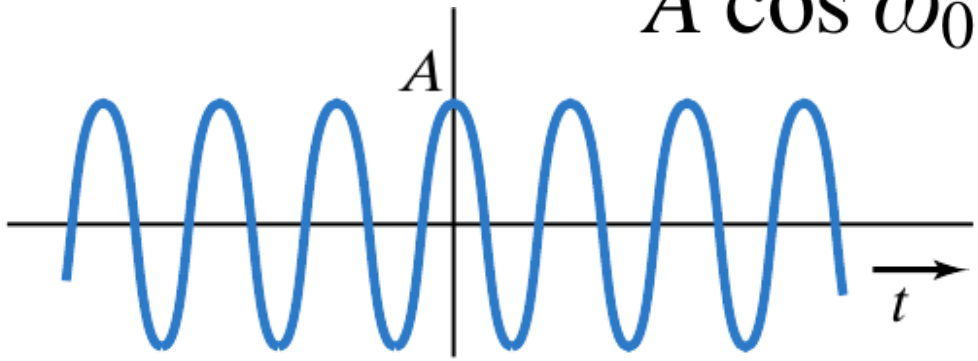



$$\begin{cases}\frac{A\pi}{b} & |\omega|<b\\ 0 & |\omega|>b\end{cases}$$



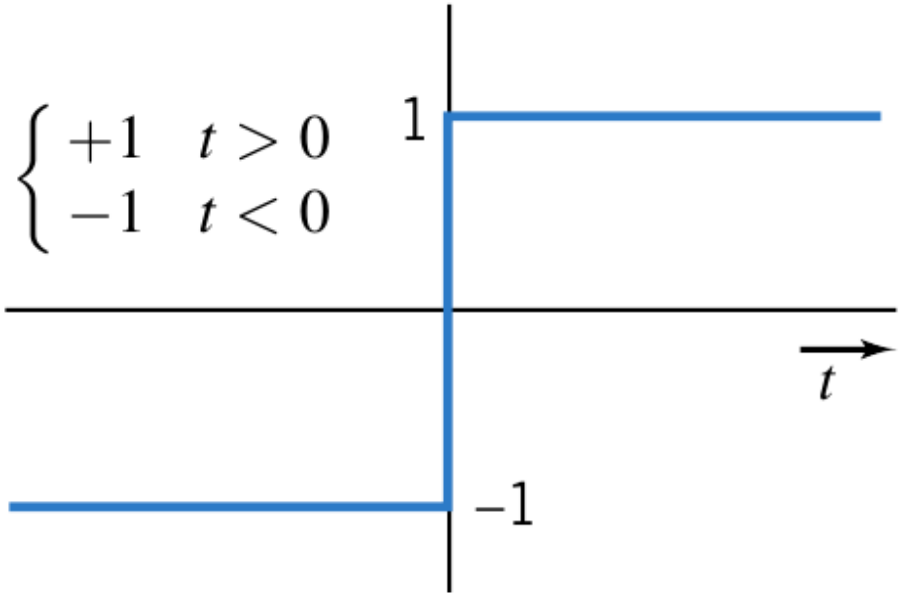

$$2\pi A\,\delta(\omega)$$

$A \cos \omega_0 t$

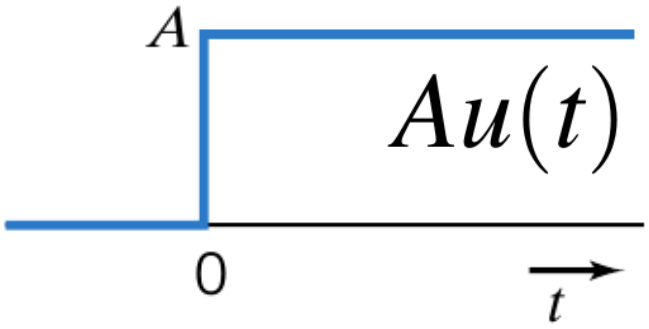



$\pi A[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$

$$f(t) = \begin{cases} +1 & t > 0 \\ -1 & t < 0 \end{cases}$$




$$\frac{2}{j\omega}$$

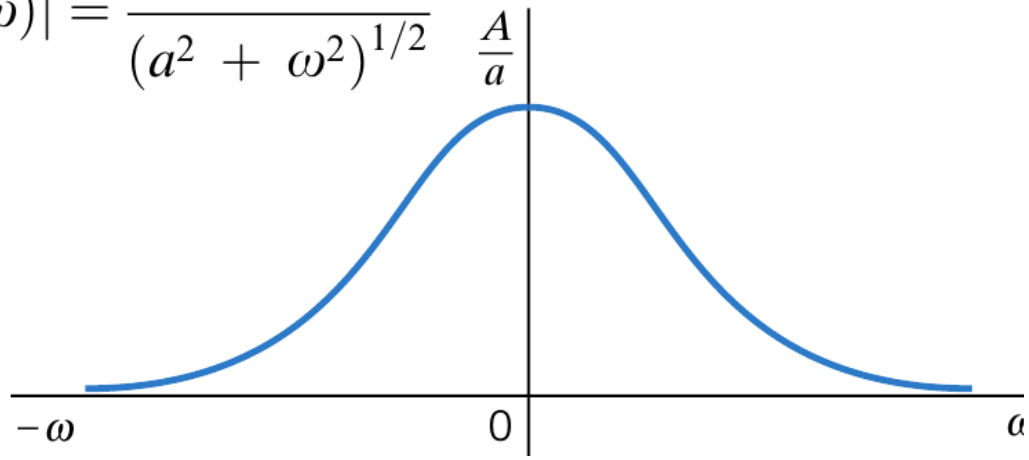


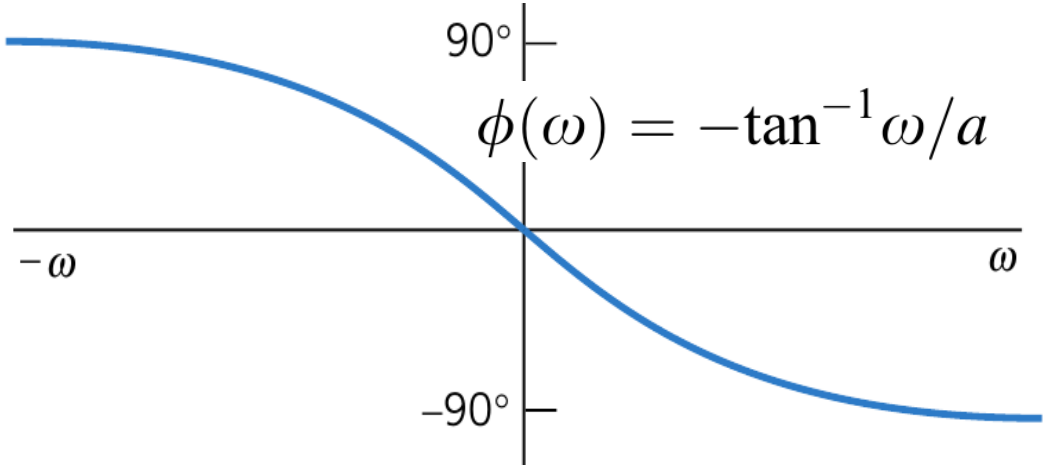

$$A\left[\pi\delta(\omega) + \frac{1}{j\omega}\right]$$

# Spectrum of Signals

- The **spectrum**, also called the **spectral density**, of a signal f (t) is its Fourier transform F(ω).
- The **Fourier spectrum** of a signal is a graph of the magnitude and phase of the Fourier transform of the signal.
- For example, when f(t) = $Ae^{-at}u(t)$, $F(\omega) = A/(a + j\omega)$
- To plot the spectrum, we calculate the magnitude & phase

$$|F(\omega)| = \frac{A}{(a^2 + \omega^2)^{1/2}}$$



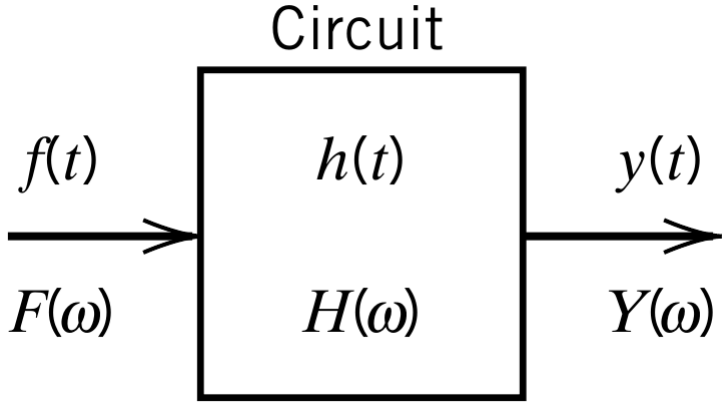
# Exercise 15.11-1

- Calculate the Fourier transform and draw the Fourier spectrum for $f(t)=A\cos\omega_0 t$ for all t.

# Convolution & Circuit Response

- A linear circuit with an impulse response h(t) and an input f(t) has response y(t)
- y(t) can be derived with:

$$y(t) = \int_{-\infty}^{\infty} h(x) f(t-x)\, dx$$




- Fourier transform of y(t):

$$\mathcal{F}[y(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x) f(t-x)\, dx\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} h(x) \int_{-\infty}^{\infty} f(t-x)\, e^{-j\omega t}\, dt\, dx$$

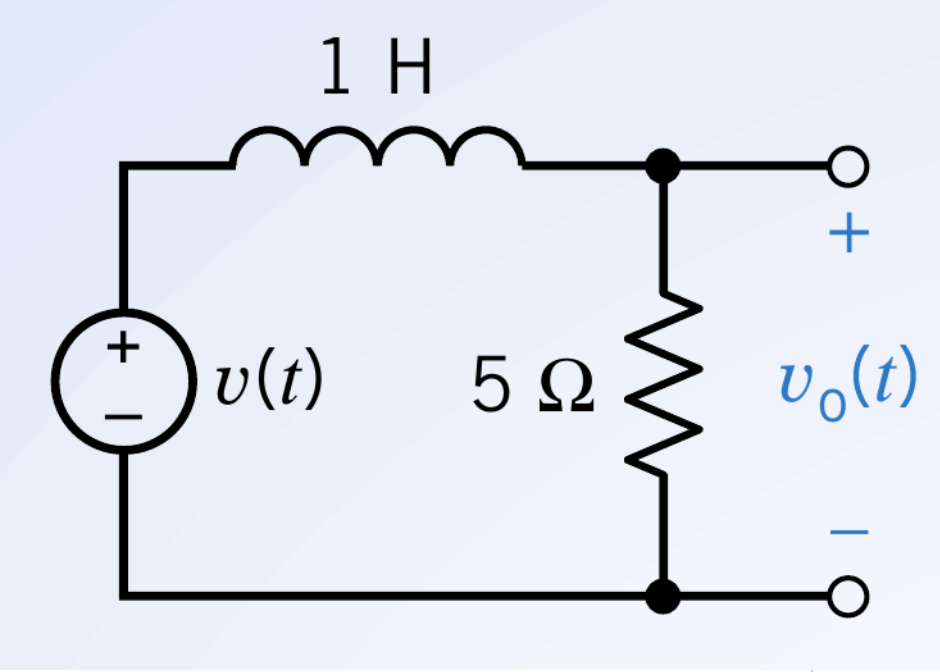
$$= \int_{-\infty}^{\infty} h(x) \int_{-\infty}^{\infty} f(u) e^{-j\omega(u+x)}\, du\, dx = \int_{-\infty}^{\infty} h(x) e^{-j\omega x}\, dx \int_{-\infty}^{\infty} f(u) e^{-j\omega u}\, du$$

- **Thus,**

$$Y(\omega) = H(\omega) F(\omega)$$

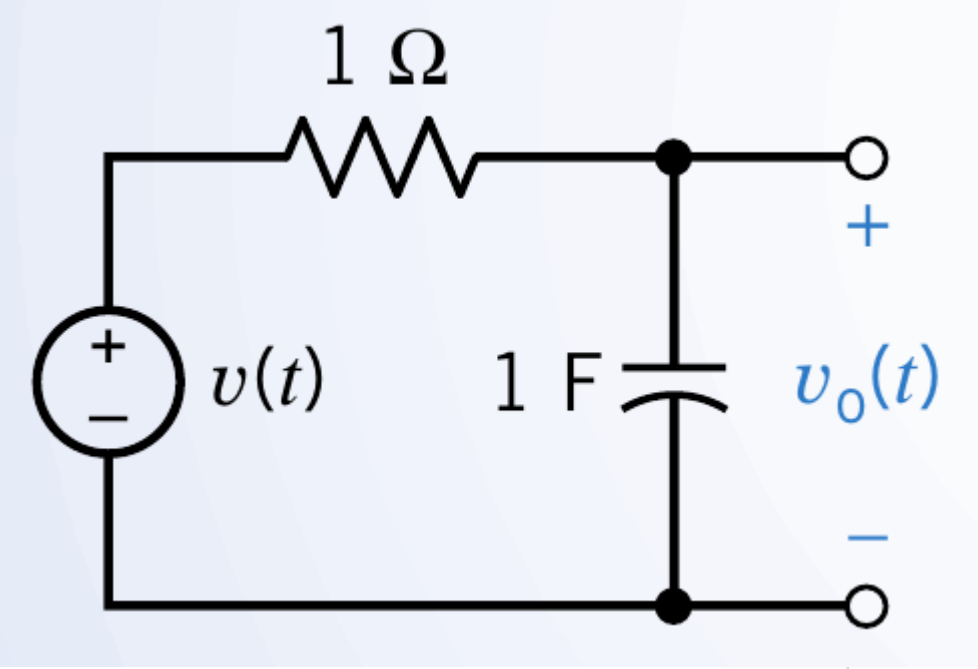
# Example 15.12-1

- $v(t) = 4e^{-2t}u(t)$
- Find $v_o(t)$

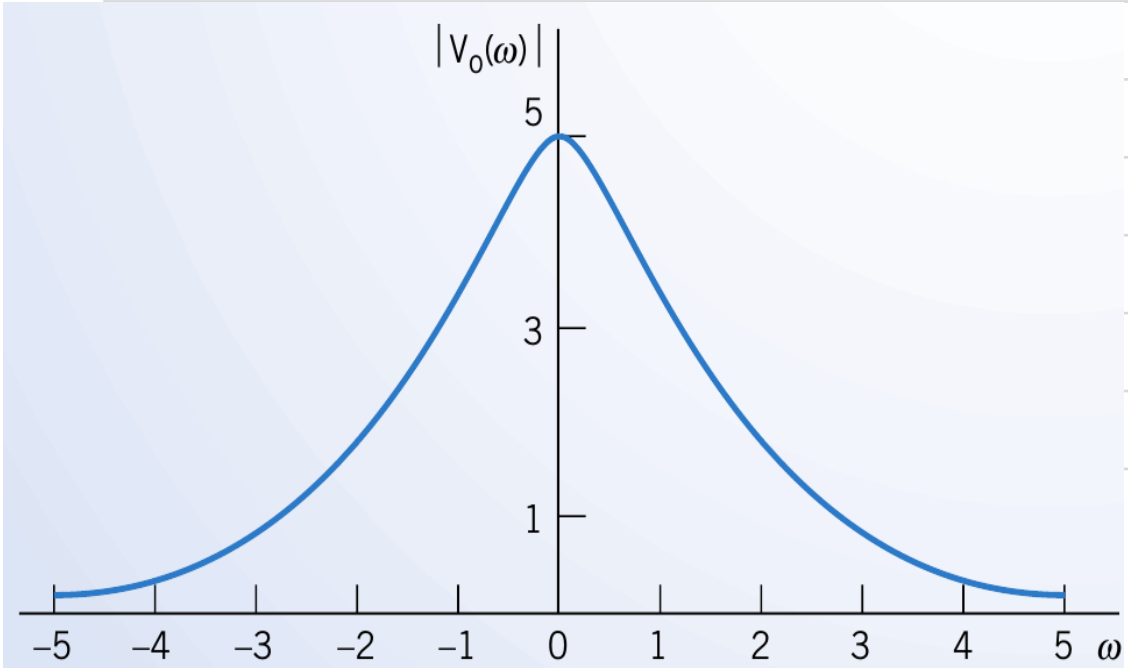


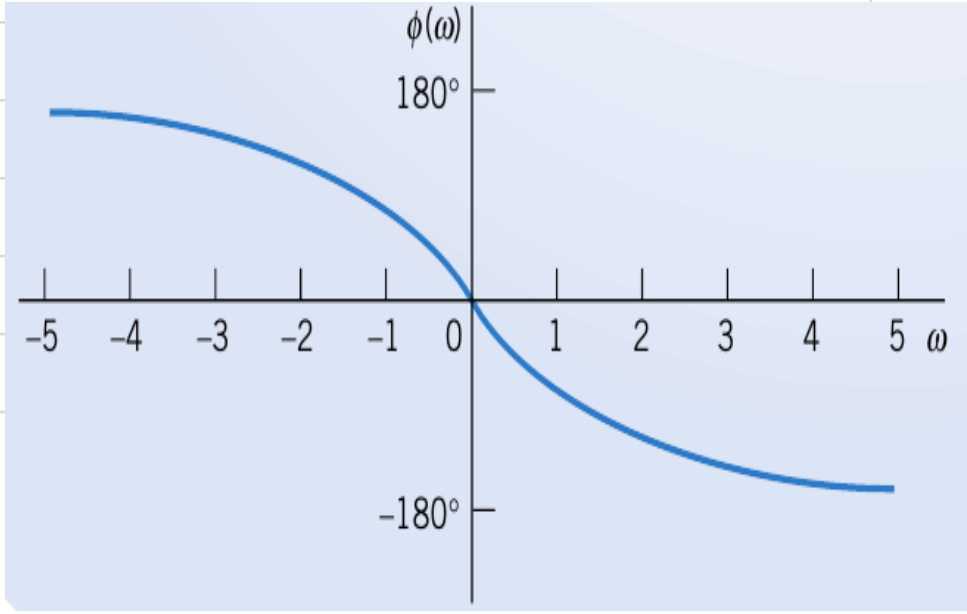
# Example 15.12-2

- $v = 10\, e^{-2t}\, u(t)$ V
- Find the spectrum of $V_o(\omega)$



# Example 15.12-2



# Fourier Transform and the Laplace Transform

- The Fourier integral:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

- The Fourier integral will converge when all the poles of F(s) lie in the left-hand s-plane. If f(t) is zero for t ≤ 0,

$$F(\omega) = F(s)\big|_{s=j\omega} \quad \text{where} \quad F(s) = \mathcal{L}[f(t)]$$

- For example, when $f(t) = Ae^{-at}u(t)$

$$F(s) = \frac{A}{s + a} \qquad F(\omega) = \frac{A}{a + j\omega}$$

# Reflection of f(t)

- When f (t) = 0 for t ≥ 0 and f (t) exists only for negative time,

$$F(\omega) = \mathcal{L}[f(-t)]|_{s=-j\omega}$$

- Note that s = -jω
- For example,

$$\begin{aligned} f(t) &= 0 \quad t \geq 0 \\ &= e^{at} \quad t < 0 \end{aligned}$$

- Then, we have $f(-t) = e^{-at} \quad t > 0$
- Therefore,

$$F(s) = \frac{1}{s + a} \xrightarrow{\text{s = -jω}} F(\omega) = \frac{1}{a - j\omega}$$

**A.** $f(t)$ nonzero for positive time only :

Step

1. $F(s) = \mathscr{L}[f(t)]$
2. $F(\omega) = F(s)\big|_{s\,=\,j\omega}$

**B.** $f(t)$ nonzero for negative time only

Step

1. $F(s) = \mathscr{L}[f(-t)]$
2. $F(\omega) = F(s)\big|_{s\,=\,-j\omega}$

**C.** $f(t)$ nonzero over all time

Step

1. $f(t) = f^{+}(t) + f^{-}(t)$
2. $F^{+}(s) = \mathscr{L}[f^{+}(t)]$
   $F^{-}(s) = \mathscr{L}[f^{-}(-t)]$
3. $F(\omega) = F^{+}(s)\big|_{s\,=\,j\omega} + F^{-}(s)\big|_{s\,=\,-j\omega}$

# Exercise 15.13-1

- Derive the Fourier transform for

$$\begin{aligned} f(t) &= te^{-at} \quad t \geq 0 \\ &= te^{at} \quad t \leq 0 \end{aligned}$$